HILBERT 90 FOR ALGEBRAS WITH CONJUGATION

PATRIK LUNDSTRÖM

Abstract. We show a version of Hilbert 90 that is valid for a large class of algebras many of which are not commutative, distributive or associative. This class contains the nth iteration of the Conway-Smith doubling procedure. We use our version of Hilbert 90 to parametrize all solutions in ordered fields to the norm one equation for such algebras.

1. Introduction

Let k be a field and suppose that K is a finite Galois field extension of k with Galois group G. Recall that the induced norm map $n : K \to k$ is defined by $n(a) = \prod_{\sigma \in G} \sigma(a)$ for all $a \in K$. The well known Theorem 90 in Hilbert's Zahlbericht [7], also known as Hilbert 90, asserts that if G is cyclic, then an $a \in K$ satisfies $n(a) = 1$ if and only if for each automorphism σ that generates G, there is a nonzero $b \in K$ satisfying $\sigma(b)a = b$. Hilbert actually only proves this in the case when K is a number field and G is of prime order. The general case is a corollary of a result by Speiser [17] which, using modern cohomological language, can be translated to the statement that $H^1(G, K^*)$ is trivial for any finite Galois field extension K of k. For more details concerning the history of Hilbert 90, see [10]. For a proof of the cohomological statement above, see e.g. Chapter VI in [9].

The impetus for this article is the observation that a version (see Theorem 1) of Hilbert 90 in degree two holds for a large class of algebras which are not necessarily commutative, distributive or associative. Namely, let K be a k-algebra. By this we mean that the following three properties hold: (i) K is a left k-vector space equipped with a multiplication and a multiplicative identity 1; (ii) k is a subset of the center of K and the additive and multiplicative structure on k equals the restriction of the additive and multiplicative structure on K; (iii) $a(bc) = (ab)c$, for $a, b, c \in K$, whenever at least one of a, b or c belongs to k. We say that K is (weak) right distributive if $(a + b)c = ac + bc$, for $a, b, c \in K$ $(a \in k)$; (weak) left distributivity is analogously defined. Furthermore, we say that K is (weak) distributive if it is both (weak) left and (weak) right distributive. Recall that K is called left (or right) alternative if $a(ab) = a^2b$ (or $(ab)b = ab^2$), for $a, b \in K$; K is called alternative if it is both left and right alternative. Let $^- : K \to K$ be a self inverse k-linear function which restricts to the identity map on k; we will refer to such a map as a conjugation. The conjugation $^- : K \to K$ is

2000 *Mathematics Subject Classification.* 11D09, 12J05, 12J15, 17A35, 17A45, 17D05.

called an involution if it is a ring antiautomorphism of K, that is, if $\overline{ab} = \overline{b}\,\overline{a}$, for $a, b \in K$. The functions $n : K \to K$ and $t : K \to K$ defined by $n(a) = \overline{a}a$ and $t(a) = a + \overline{a}$, for $a \in K$, respectively, will be referred to as the norm and trace on K. We say that an element $a \in K$ is imaginary if $t(a) = 0$. We say that n is anisotropic if $n(a) \neq 0$, for nonzero $a \in K$, and we say that n is multiplicative if $n(ab) = n(a)n(b)$, for $a, b \in K$. Furthermore, we say that n is symmetric if $n(\overline{a}) = n(a)$, for $a \in K$.

In Section 2, we show the following version of Hilbert 90 for algebras with conjugation.

Theorem 1. *Suppose that K is a weak distributive left alternative k-algebra equipped with a conjugation making the trace k-valued and the norm anisotropic and k-valued. If K has a nonzero imaginary element, then an $a \in K$ satisfies $n(a) = 1$ if and only if there is a nonzero $b \in K$ satisfying $\overline{b}a = b$.*

There are lots of algebras with conjugation. The most well known examples are the ones constructed via the Cayley-Dickson doubling process (for more details, see e.g. [2] or [14]). Namely, suppose that K is a k-algebra equipped with a conjugation and that we have chosen a $D \in k$. Then there is a conjugation and a multiplication on the k-vector space $K \times K$ defined by the relations $\overline{(a, b)} = (\overline{a}, -b)$ and $(a, b)(c, d) = (ac + Dd\overline{b}, \overline{a}d + cb)$, respectively, for $a, b, c, d \in K$. The resulting structure, called the Cayley-Dickson double of K with respect to D, is in this article denoted $K \times_D K$; it is a k-algebra with a nonzero multiplicative identity $(1, 0)$ and a nonzero imaginary element $(0, 1)$. Starting with the real numbers equipped with the trivial conjugation this process successively, with $D = -1$ at each step, yields the complex numbers, the quaternions, the octonions and the sedenions. However, their properties keep degrading; the complex numbers are not ordered, the quaternions are not commutative, the octonions are not associative and the sedenions contain zero divisors and hence do not have a multiplicative norm.

The absence of a multiplicative norm for the sedenions is implied by the well known Hurwitz's theorem [8] which states that any real distributive algebra equipped with a multiplicative norm is necessarily isomorphic to either the real numbers, the complex numbers, the quaternions or the octonions. The existence of these four algebras is equivalent to that of N-square identities $(x_1^2 + \cdots + x_N^2)(y_1^2 + \cdots + y_N^2) = (z_1^2 + \cdots + z_N^2)$ for $N = 1, 2, 4, 8$ in which the z_k are bilinear functions of the x_i and y_j. In 1967 Pfister [12, 13], quite surprisingly, showed the existence of such N-square identities for any N which is a power of two provided that the z_k are rational functions of the x_i and y_j. Conway [3] has pointed out that such identities can be produced by various modifications of the Cayley-Dickson procedure. J. D. H. Smith [15] has proposed a similar doubling procedure, which, however, does not behave well algebraically; e.g. when applied to the quaternions it does not yield the octonions. Here, we will be concerned with a variation of Conway's procedure proposed by W. D. Smith in his extensive article [16]. Since it seems that this approach was developed, at least partially, in collaboration with Conway, we will refer to it as Conway-Smith doubling (for more details concerning priority, see Section 32 in loc. cit.). The Conway-Smith process starting from the real numbers, not only produces the complex numbers, the quaternions and the octonions, but keeps going to yield 2^n-dimensional algebraical structures with multiplicative norms for any nonnegative integer n; Smith calls these structures the 2^n-ons. This procedure supposes that K is a division algebra over k. By this we mean that to each nonzero $a \in K$ there is a unique $b \in K$ satisfying $ab = ba = 1$; in that case we will write $b = a^{-1}$. The Conway-Smith double of K with respect to $D \in k$ (Smith loc. cit. always keeps $D = -1$), in this article denoted $K \times^D K$, is the k-vector space $K \times K$ equipped with the same conjugation as for the Cayley-Dickson double but with a multiplication defined, for all $a, b, c, d \in K$, by

$$(a,b)(c,d) = \begin{cases} \left(ac + D\overline{b\overline{d}}, \overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}}} \right), \text{ if } b \neq 0, \\ \\ (ac, \overline{a}d), \text{ otherwise.} \end{cases}$$

The k-algebra $K \times^D K$ has a multiplicative identity $(1, 0)$ and a nonzero imaginary element $(0, 1)$.

In Section 3 and Section 4, we investigate when Theorem 1 holds for the Cayley-Dickson and Conway-Smith doubling processes (see Corollaries 1, 2 and 3). As an application, we parametrize explicitly the solutions to the norm one equation for the nth iteration of the Conway-Smith doubling of k (see Remark 4). We are also able to show that this works for the first, second and third Cayley-Dickson doubling of k (see Remark 3). Along the way, we give an alternative proof of norm multiplicativity of the 2^n-ons (see Corollary 5) and for the first, second and third Cayley-Dickson doubling (see Corollary 3). This and a few other results of ours can be found in Smith's article [16] in the case when the doubling is defined by $D = -1$. However, whereas Smith loc. cit. often resorts to arguments depending on (quasi) matrix representations in his proofs, we use algebraical properties of the doubling process alone. For the convenience of the reader, we have tried to make our presentation as self-contained as possible.

2. Algebras with Conjugation

In the end of this section, we show Theorem 1. Along the way, and for use in the following sections, we show some results concerning general algebras with conjugation (see Propositions 1-4). For the rest of the article, we suppose that K is a k-algebra equipped with a conjugation $^- : K \to K$.

Proposition 1. *Suppose that K is weak distributive and equipped with a nonzero imaginary element. If an $a \in K$ satisfies $n(a) = 1$, then there is a nonzero $b \in K$ satisfying $\overline{b}a = b$.*

Proof. Suppose that $n(a) = 1$ for some $a \in K$. We consider two cases.

Case 1: $a = -1$. Let b be a nonzero imaginary element of K. Then $\overline{b} = -b$ and hence, by left weak distributivity, we get that $\overline{b}a = (-b)(-1) = (-b)(-1) + 0 = (-b)(-1) + (-b) + b = (-b)(-1+1) + b = (-b) \cdot 0 + b = b$.

Case 2: $a \neq -1$. Let $b = a + 1$. Then, by weak right distributivity, we get that $\overline{b}a = \left(\overline{a+1}\right) a = (\overline{a} + 1)\, a = \overline{a}a + a = n(a) + a = 1 + a = b$. □

Proposition 2. *If K is weak right distributive and the trace and norm on K are k-valued, then K is left alternative if and only if $\overline{a}(ab) = n(a)b$ for all $a, b \in K$. In that case, if the norm is anisotropic and symmetric, then K is a division algebra with $a^{-1} = \overline{a}n(a)^{-1}$ for all nonzero $a \in K$.*

Proof. Suppose that K is weak right distributive and that the trace and norm are k-valued. Take $a, b \in K$. First we show the "only if" part of the proof. Suppose that K is alternative. Then

$$\overline{a}(ab) = (t(a) - a)(ab) = t(a)ab - a(ab) = t(a)ab - a^2b =$$

$$= t(a)ab - ((t(a) - \overline{a})a)b = t(a)ab - (t(a)a - \overline{a}a)b =$$

$$= t(a)ab - (t(a)a - n(a))b = t(a)ab - (t(a)ab - n(a)b) = n(a)b.$$

Therefore, $\overline{a}(ab) = n(a)b$. Now we show the "if" part of the proof. Suppose that $\overline{a}(ab) = n(a)b$. Then

$$a(ab) = (t(a) - \overline{a})(ab) = t(a)ab - \overline{a}(ab) = t(a)ab - n(a)b =$$

$$= (t(a)a - n(a))b = (t(a)a - \overline{a}a)b = ((t(a) - \overline{a})a)b = a^2b.$$

Therefore, $a(ab) = a^2b$. Now suppose that the norm also is anisotropic and symmetric and that a is nonzero. Then

$$\overline{a}n(a)^{-1}a = \overline{a}an(a)^{-1} = n(a)n(a)^{-1} = 1$$

and

$$a\overline{a}n(a)^{-1} = \overline{\overline{a}}\,\overline{a} = n(\overline{a})n(a)^{-1} = n(a)n(a)^{-1} = 1.$$

Therefore, $\overline{a}n(a)^{-1}$ is a multiplicative inverse of a. Suppose that $ab = 1$. Then, by the above, we get that

$$b = n(a)bn(a)^{-1} = \overline{a}(ab)n(a)^{-1} = \overline{a} \cdot 1 \cdot n(a)^{-1} = \overline{a}n(a)^{-1}.$$

Therefore, the multiplicative inverse of a is unique. □

Proposition 3. *Suppose that the trace on K is k-valued. (a) If K is weak right distributive, then $t(a)b = ab + \overline{a}b$ for all $a, b \in K$; (b) If K is weak distributive, then the norm on K is symmetric; (c) If K is weak distributive and the norm on K is k-valued, then the conjugation on K respects squares; (d) If K is left alternative, weak distributive and the norm is k-valued, then the norm respects squares.*

Proof. Take $a, b \in K$ and suppose that the trace on K is k-valued.

(a) If K is weak right distributive, then

$$t(a)b = ab + t(a)b - ab = ab + (t(a) - a)b = ab + \overline{a}b.$$

(b) If K is weak distributive, then

$$n(\overline{a}) = \overline{\overline{a}}\,\overline{a} = a\overline{a} = a(t(a) - a) = t(a)a - a^2 = (t(a) - a)a = \overline{a}a = n(a).$$

(c) If K is weak distributive and the norm on K is k-valued, then

$$\overline{a^2} = \overline{(t(a) - \overline{a})a} = \overline{t(a)a - n(a)} = t(a)\overline{a} - n(a) =$$

$$= \overline{a}t(a) - \overline{a}a = \overline{a}(t(a) - a) = \overline{a}^2.$$

(d) By Proposition 2, left alternativity of K and (c), we get that

$$n(a^2) = \overline{a^2}a^2 = \overline{a}^2 a^2 = \overline{a}(\overline{a}a^2) = \overline{a}(n(a)a) = n(a)\overline{a}a = n(a)^2.$$

□

In many cases the norm not only respects squares, but is in fact multiplicative. In the sequel, we will use the following result.

Proposition 4. *Suppose that K is weak right distributive and left alternative with k-valued trace and norm. If there is a k-linear map $T : K \to K$ which restricts to the identity map on k and satisfies*

$$T\left(\overline{a(bc)}\right) = T\left(\overline{c}\left(\overline{b}\overline{a}\right)\right) \tag{1}$$

for all $a, b, c \in K$, then the norm on K is multiplicative.

Proof. Take $a, b \in K$. Then, by (1) and Proposition 2, we get that

$$n(ab) = T(n(ab)) = T\left(\overline{ab}\;ab\right) = T\left(\overline{ab}\left(\overline{\overline{a}}\,\overline{\overline{b}}\right)\right) = T\left(\overline{\overline{b}\left(\overline{a}(ab)\right)}\right) =$$

$$= T\left(\overline{\overline{b}n(a)b}\right) = n(a)T\left(\overline{\overline{b}b}\right) = n(a)T\left(\overline{n(b)}\right) = n(a)n(b).$$

□

Remark 1. If K is distributive, alternative and equipped with an involution making the trace and the norm k-valued, then, by a classical argument, the norm on K is multiplicative. Namely, by a theorem of Artin (see Theorem 3.1 in [14]), every subalgebra generated by two elements of a distributive alternative algebra is associative. Therefore, for any $a, b \in K$, we get that

$$n(ab) = \overline{(ab)}(ab) = (\overline{b}\overline{a})(ab) = (t(b) - b)(t(a) - a)ab =$$

$$= \overline{b}(\overline{a}a)b = \overline{b}n(a)b = n(a)\overline{b}b = n(a)n(b).$$

However, since many of the structures we study in the sequel possess conjugations which are not in general involutions, Proposition 4 is still motivated.

Proof of Theorem 1. The ”only if” part of the claim follows from Proposition 1. Now we show the ”if” part of the claim. Suppose that $\overline{b}a = b$ for some nonzero $a, b \in K$. By Proposition 2 and Proposition 3(b), we get that $n(b)a = n(\overline{b})a = b(\overline{b}a) = b^2$. By taking norms we get, by Proposition 3(d), that $n(b)^2 n(a) = n(b)^2$. Since $n(b) \neq 0$, this implies that $n(a) = 1$. □

3. Cayley-Dickson doubling

In this section, we investigate when Theorem 1 holds for algebras generated by the Cayley-Dickson doubling process (see Corollaries 1 and 2). For the rest of the article, we assume that D is an element of k.

Proposition 5. *(a) K is weak right distributive if and only if $K \times_D K$ is weak right distributive; (b) $K \times_D K$ is (weak) left distributive if and only if $K \times_D K$, and hence K, is (weak) distributive; (c) $K \times_D K$ is right distributive if and only if $K \times_D K$, and hence K, is distributive; (d) K is weak distributive if and only if $K \times_D K$ is weak distributive.*

Proof. (a) The ”if” part of the claim is clear since K is contained in $K \times_D K$. Now we show the ”only if” part of the claim. Suppose that K is weak right distributive. Take $a_1 \in k$ and $a_2, b, c, d \in K$. Then

$$((a_1, 0) + (a_2, b))\,(c, d) = (a_1 + a_2, b)(c, d) = ((a_1 + a_2)c + Dd\overline{b}, (\overline{a_1 + a_2})d + cb) =$$

$$= ((a_1 + a_2)c + Dd\overline{b}, (\overline{a}_1 + \overline{a}_2)d + cb) = (a_1 c + a_2 c + Dd\overline{b}, \overline{a}_1 d + \overline{a}_2 d + cb) =$$

$$= (a_1 c, \overline{a}_1 d) + (a_2 c + Dd\overline{b}, \overline{a}_2 d + cb) = (a_1, 0)(c, d) + (a_2, b)(c, d).$$

Therefore $K \times_D K$ is weak right distributive.

(b) Suppose that $K \times_D K$ is left distributive. Since K is contained in $K \times_D K$, we get that K is left distributive. Now we show that K is right distributive. Take $a, b, c \in K$. Then

$$(0, 0) = (0, c)(a + b, 0) - (0, c)(a, 0) - (0, c)(b, 0) = (0, (a + b)c - ac - bc).$$

Therefore $0 = (a + b)c - ac - bc$. Now we show that $K \times_D K$, and hence K, is right distributive. Take $a, b, c, d, e, f \in K$. Then

$$((a, b) + (c, d))(e, f) = (a + c, b + d)(e, f) =$$

$$= \big((a + c)e + Df(\overline{b + d}), (\overline{a + c})f + e(b + d)\big)$$

$$= \big(ae + ce + Df(\overline{b} + \overline{d}), (\overline{a} + \overline{c})f + eb + ed\big)$$

$$= \big(ae + ce + Df\overline{b} + Df\overline{d}, \overline{a}f + \overline{c}f + eb + ed\big)$$

$$= \big(ae + Df\overline{b}, \overline{a}f + eb\big) + \big(ce + Df\overline{d}, \overline{c}f + ed\big) = (a, b)(e, f) + (c, d)(e, f).$$

The weak part of (b) is proved in a similar way.

(c) This can be proved in a fashion analogous to (b).

(d) This follows from (a) and (b). □

Corollary 1. *Suppose that k is equipped with the identity conjugation. If we for each positive integer i choose $D_i \in k$ and recursively define the k-algebras k_i, for $i \geq 0$, by $k_0 = k$ and $k_i = k_{i-1} \times_{D_i} k_{i-1}$, for $i \geq 1$, then, for all $i \geq 1$, if an $a \in k_i$ satisfies $n(a) = 1$, then there is a nonzero $b \in k_i$ satisfying $\overline{b}a = b$.*

Proof. This follows immediately from Propositions 1 and 5. □

Proposition 6. *(a) The conjugation on K is an involution if and only if the conjugation on $K \times_D K$ is an involution; (b) The trace on K is k-valued if and only if the trace on $K \times_D K$ is k-valued; (c) The norm on K is symmetric if and only if the norm on $K \times_D K$ is symmetric. In that case (i) the norm on K is k-valued if and only if the norm on $K \times_D K$ is k-valued; (ii) the norm on $K \times_D K$ is anisotropic if and only if the norm on K is anisotropic and D is an element of k not belonging to the set of quotients $n(a)/n(b)$, for $a \in K$ and nonzero $b \in K$.*

Proof. All the "only if" parts of the proof follow from the fact that K is contained in $K \times_D K$. Now we show the "if" parts of the proof. Take $a, b, c, d \in K$.

(a) Suppose that the conjugation on K is an involution. Then we get that

$$\overline{(a,b)(c,d)} = \overline{(ac + Dd\overline{b}, \overline{a}d + cb)} = \left(\overline{ac} + D\overline{d\overline{b}}, -\overline{a}d - cb\right) =$$

$$= \left(\overline{c}\ \overline{a} + D\overline{\overline{b}}\ \overline{d}, -\overline{a}d - cb\right) = \left(\overline{c}\ \overline{a} + D(-b)(\overline{-d}), \overline{\overline{c}}(-b) + \overline{a}(-d)\right) =$$

$$= (\overline{c}, -d)(\overline{a}, -b) = \overline{(c,d)}\ \overline{(a,b)}.$$

Therefore the conjugation on $K \times_D K$ is an involution.

(b) Suppose that the trace on K is k-valued. Then we get that

$$t(a,b) = (a,b) + \overline{(a,b)} = (a,b) + (\overline{a}, -b) = (a + \overline{a}, b - b) = (t(a), 0)$$

which belongs to k.

(c) Suppose that the norm on K is symmetric. Then we get that

$$n((a,b)) = \overline{(a,b)}(a,b) = (\overline{a}, -b)(a,b) = (\overline{a}a + Db(-\overline{b}), \overline{\overline{a}}b + a(-b)) =$$

$$= (n(a) + D(-b)\overline{b}, ab - ab) = (n(\overline{a}) + D(-b)\overline{b}, 0) =$$

$$= (\overline{\overline{a}}\ \overline{a} + D(-b)\overline{b}, \overline{a}(-b) + \overline{a}b) = (a\overline{a} + D(-b)\overline{b}, \overline{a}(-b) + \overline{a}b)) =$$

$$= (a,b)(\overline{a}, -b) = (a,b)\overline{(a,b)} = \overline{\overline{(a,b)}}(a,b) = n\left(\overline{(a,b)}\right).$$

Therefore, the norm on $K \times_D K$ is symmetric. Furthermore, the above calculation shows that $n((a,b)) = n(a) - Dn(b)$. From this equality, (i) and (ii) follow immediately. □

Proposition 7. *If K is left distributive, weak right distributive and both the trace and the norm on K are k-valued, then K is associative if and only if $K \times_D K$ is left alternative.*

Proof. Suppose that $K \times_D K$, and hence K, is left alternative. Take $a, b, c \in K$. Then, by Proposition 3(a)(b), we get that

$$(0,0) = (a,c)^2(b,0) - (a,c)((a,c)(b,0)) =$$
$$= (a^2 + Dc\overline{c}, \overline{a}c + ac)(b,0) - (a,c)(ab,bc)$$
$$= (a^2 + Dn(c), t(a)c)(b,0) - (a^2b + D(bc)\overline{c}, \overline{a}(bc) + (ab)c)$$
$$= (a^2b + Dn(c)b, t(a)bc) - (a^2b + Dbn(c), \overline{a}(bc) + (ab)c)$$
$$= (0, (t(a) - \overline{a})bc - (ab)c) = (0, a(bc) - (ab)c).$$

Therefore $a(bc) - (ab)c = 0$ and hence K is associative.

Suppose that K is associative. Take $a, b, c, d \in K$. Then, by Proposition 3(c), we get that

$$(a,b)^2(c,d) = (a^2 + Dn(b), t(a)b)(c,d) =$$
$$= \left((a^2 + Dn(b))c + Dt(a)d\overline{b}, \left(\overline{a^2 + Dn(b)}\right) d + t(a)cb\right)$$
$$= (a^2c + Dn(b)c + Dt(a)d\overline{b}, \overline{a}^2 d + Dn(b)d + t(a)cb)$$

and, by Proposition 3(a), we get that

$$(a,b)((a,b)(c,d)) = (a,b)(ac + Dd\overline{b}, \overline{a}d + cb) =$$
$$= (a(ac + Dd\overline{b}) + D(\overline{a}d + cb)\overline{b}, \overline{a}(\overline{a}d + cb) + (ac + Dd\overline{b})b)$$
$$= (a^2c + Dad\overline{b} + D\overline{a}d\overline{b} + Dcn(b), \overline{a}^2 d + \overline{a}cb + acb + Ddn(b))$$
$$= (a^2c + Dt(a)d\overline{b} + Dcn(b), \overline{a}^2 d + Ddn(b) + t(a)cb).$$

Therefore $(a,b)^2(c,d) = (a,b)((a,b)(c,d))$ and hence $K \times_D K$ is alternative. □

Recall that k is called ordered if it is equipped with a total order $\leq$ satisfying the following two properties for all $a, b, c \in k$: (i) if $a \leq b$, then $a + c \leq b + c$; (ii) if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$. An element $a \in k$ is called positive if $0 \leq a$ and $0 \neq a$. Negative elements of k are analogously defined.

Corollary 2. *Suppose that the k-algebras k_1, k_2 and k_3 are defined as in Corollary 1. Then an $a \in k_i$ satisfies $n(a) = 1$ if and only if there is a nonzero $b \in k_i$ with $\overline{b}a = b$ if (a) $i = 1$ and D_1 is not a square in k or (b) $i = 1, 2, 3$ whenever k is an ordered field and D_1, D_2 and D_3 are negative.*

Proof. This follows immediately from Theorem 1 and Propositions 5-7. □

Remark 2. It is not clear to the author at present whether the conclusion of Corollary 2 holds for $i \geq 4$ since the techniques we have used depend on the fact that the algebras are left alternative; it is well known that this is not satisfied for algebras containing the sedenions, that is, if $i \geq 4$.

Remark 3. By folklore, Hilbert 90 can be used to parametrize solutions to the norm one equation. Now we apply this idea to the k-algebras k_1, k_2 and k_3 as defined in Corollary 1. Put $C_i = -D_i$ for $i = 1, 2, 3$. Given $x \in k_i$ and $1 \leq i \leq 3$, then, by the proof of Theorem 1, the equality $n(x) = 1$ holds if and only if there is a nonzero $s \in k_i$ with

$$x = \frac{s^2}{n(s)} = \frac{2s_1^2 - n(s)}{n(s)} + \frac{2s_1(s - s_1)}{n(s)} \tag{2}$$

where s_1 is the k-part of s. Thus, by calculating the norm of a general element of k_i, for $i = 1, 2, 3$, we can parametrize the solutions to the norm one equation. We now make this parametrization explicit in the cases $i = 1, 2, 3$ separately.

If D_1 is not a square in k, then, by (2), we get that $x_1, x_2 \in k$ satisfy

$$x_1^2 + C_1 x_2^2 = 1$$

if and only if there are $s_1, s_2 \in k$, not both zero, such that

$$x_1 = \frac{s_1^2 - C_1 s_2^2}{s_1^2 + C_1 s_2^2} \qquad \text{and} \qquad x_2 = \frac{2 s_1 s_2}{s_1^2 + C_1 s_2^2}$$

In particular, this holds if k is an ordered field and C_1 is positive.

If k is an ordered field and C_1 and C_2 are positive, then, by (2), we get that $x_i \in k$, for $i = 1, 2, 3, 4$, satisfy

$$x_1^2 + C_1 x_2^2 + C_2 x_3^2 + C_1 C_2 x_4^2 = 1$$

if and only if there are $s_1, s_2, s_3, s_4 \in k$, not all zero, such that

$$x_1 = \frac{s_1^2 - C_1 s_2^2 - C_2 s_3^2 - C_1 C_2 s_4^2}{s_1^2 + C_1 s_2^2 + C_2 s_3^2 + C_1 C_2 s_4^2}$$

and

$$x_i = \frac{2 s_1 s_i}{s_1^2 + C_1 s_2^2 + C_2 s_3^2 + C_1 C_2 s_4^2}$$

for $i = 2, 3, 4$.

If k is an ordered field and C_1, C_2 and C_3 are positive, then, by (2), we get that $x_i \in k$, for $i = 1, \ldots, 8$, satisfy

$$x_1^2 + C_1 x_2^2 + C_2 x_3^2 + C_1 C_2 x_4^2 + C_3 x_5^2 + C_1 C_3 x_6^2 + C_2 C_3 x_7^2 + C_1 C_2 C_3 x_8^2 = 1$$

if and only if there are $s_i \in k$, for $1 \leq i \leq 8$, not all zero, such that

$$x_1 = \frac{s_1^2 - C_1 s_2^2 - C_2 s_3^2 - C_1 C_2 s_4^2 - C_3 s_5^2 - C_1 C_3 s_6^2 - C_2 C_3 s_7^2 - C_1 C_2 C_3 s_8^2}{s_1^2 + C_1 s_2^2 + C_2 s_3^2 + C_1 C_2 s_4^2 + C_3 s_5^2 + C_1 C_3 s_6^2 + C_1 C_3 s_7^2 + C_1 C_2 C_3 s_8^2}$$

and

$$x_i = \frac{2 s_1 s_i}{s_1^2 + C_1 s_2^2 + C_2 s_3^2 + C_1 C_2 s_4^2 + C_3 s_5^2 + C_1 C_3 s_6^2 + C_2 C_3 s_7^2 + C_1 C_1 C_3 s_8^2}$$

for $2 \leq i \leq 8$.

Remark 4. The technique used in Remark 3 to parametrize the solutions to the norm one equation for quadratic extensions, and, in particular, to use this to parametrize pythagorean triples, can be found in several places in the literature, see e.g. [4], [6], [11] or [18]. A similar discussion for the biquadratic case can be found in [5]. See also the end of Remark 5.

We end this section with a digression on norm multiplicativity of Cayley-Dickson doubles.

Proposition 8. *Suppose that K is associative and left distributive and there is a k-linear map $T : K \to K$ satisfying the following three conditions (i) $T(\overline{a}) = T(a)$; (ii) $T(ab) = T(ba)$; (iii) $T\left(\overline{a(bc)}\right) = T\left(\overline{c}\left(\overline{b}\overline{a}\right)\right)$ for all $a, b, c \in K$. If we extend the map T to $K \times_D K$ by the relation $T((a, b)) = T(a)$, for $a, b \in K$, then (i), (ii) and (iii) hold for all $a, b, c \in K \times_D K$. In that case, if K is weak right distributive, then the norm on $K \times_D K$ is multiplicative.*

Proof. Take $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ in $K \times_D K$.

First we show (i): $T(\overline{a}) = T(\overline{a}_1, -a_2) = T(\overline{a}_1) = T(a_1) = T(a)$.

Next we show (ii): $T(ab) = T(a_1b_1 + Db_2\overline{a}_2) = T(a_1b_1) + DT(b_2\overline{a}_2) = T(b_1a_1) + DT\left(\overline{b_2\overline{a}_2}\right) = T(b_1a_1) + DT(a_2\overline{b}_2) = T(ba)$.

Finally, we show (iii). First note that left distributivity in combination with (ii) implies that

$$T((x + y)z) = T(z(x + y)) = T(zx + zy) =$$
$$= T(zx) + T(zy) = T(xy) + T(yz) = T(xy + yz)$$

for all $x, y, z \in K$. Therefore, by (i), we get that

$$T\left(\overline{a(bc)}\right) = T\left(a_1\left(b_1c_1 + Dc_2\overline{b}_2\right)\right) + DT\left(\left(\overline{b}_1c_2 + c_1b_2\right)\overline{a}_2\right) =$$
$$= T\left(\overline{a_1(b_1c_1)}\right) + DT\left(\overline{a_1\left(c_2\overline{b}_2\right)}\right) + DT\left(\overline{(\overline{b}_1c_2)\overline{a}_2}\right) + DT\left(\overline{(c_1b_2)\overline{a}_2}\right)$$
$$= T\left(\overline{c}_1\left(\overline{b}_1\overline{a}_1\right)\right) + DT\left(b_2\left(\overline{c}_2\,\overline{a}_1\right)\right) + DT\left(a_2\left(\overline{c}_2b_1\right)\right) + DT\left(a_2\left(\overline{b}_2\overline{c}_1\right)\right).$$

Likewise

$$T\left(\overline{c}\left(\overline{b}\overline{a}\right)\right) = T\left(\overline{c}_1\left(\overline{b}_1\overline{a}_1\right)\right) + DT\left(\overline{c}_1\left(a_2\overline{b}_2\right)\right) + DT\left((b_1a_2)\overline{c}_2\right) + DT\left(\left(\overline{a}_1b_2\right)\overline{c}_2\right)$$

which, by (ii) and associativity of K, equals $T\left(\overline{a(bc)}\right)$.

The last part follows from the above and Propositions 4 and 7. □

Corollary 3. *The norms on the k-algebras k_1, k_2 and k_3, as defined in Corollary 1, are multiplicative.*

Proof. Suppose that we define the k-linear map $T_3 : k_3 \to k$, as the projection on k. By restriction, this induces k-linear maps $T_2 : k_2 \to k$, $T_1 : k_1 \to k$ and $T_0 : k_0 \to k$. Since obviously $T_0 = \mathrm{id}_k$ satisfies (i), (ii) and (iii) from Proposition 8, it follows that the same is true for T_1, T_2 and T_3. Therefore, by Propositions 4, 5 and 7, the norm is multiplicative on k_1, k_2 and k_3. □

4. Conway-Smith doubling

In this section, we investigate when Theorem 1 holds for algebras generated by the Conway-Smith doubling process.

Proposition 9. *If K is left alternative, left distributive, weak right distributive, and has k-valued trace and anisotropic k-valued norm, then $K \times^D K$ is left alternative, left distributive, weak right distributive and has k-valued trace and norm. In that case, if D does not belong to the set of quotients $n(a)/n(b)$, for $a \in K$ and nonzero $b \in K$, then $K \times^D K$ has anisotropic norm.*

Proof. Take $a, b, c, d, e, f \in K$. First we show that $K \times^D K$ is left distributive. Case 1: $b = 0$. Then

$$(a,b)((c,d)+(e,f)) = (a,b)(c+e,d+f) = (a,0)(c+e,d+f) =$$
$$= (a(c+e), \overline{a}(d+f)) = (ac+ae, \overline{a}d+\overline{a}f) = (ac,\overline{a}d) + (ae,\overline{a}f) =$$
$$= (a,0)(c,d) + (a,0)(e,f) = (a,b)(c,d) + (a,b)(e,f).$$

Case 2: $b \neq 0$. Then

$$(a,b)((c,d)+(e,f)) = (a,b)(c+e,d+f) =$$
$$= \left(a(c+e) + D\overline{b(\overline{d+f})}, \overline{\overline{b}(\overline{c+e})} + \overline{\overline{b}\,\overline{\overline{a}\,\overline{\overline{b^{-1}}(\overline{d+f})}}}\right)$$
$$= \left(ac + D\overline{b\overline{d}}, \overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}}}\right) + \left(ae + D\overline{b\overline{f}}, \overline{\overline{b}\,\overline{e}} + \overline{\overline{b}\,\overline{\overline{a}\,\overline{\overline{b^{-1}}\,\overline{f}}}}\right)$$
$$= (a,b)(c,d) + (a,b)(e,f).$$

Now we show that $K \times^D K$ is weak right distributive. If $a \in k$, then

$$((a,0)+(c,d))(e,f) = (a+c,d)(e,f) =$$
$$= \left((a+c)e + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{\overline{d}\,\overline{(\overline{a+c})\overline{\overline{d^{-1}}\,\overline{f}}}}\right)$$
$$= \left(ae + ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{\overline{d}\,\overline{(\overline{a}+\overline{c})\overline{\overline{d^{-1}}\,\overline{f}}}}\right)$$
$$= \left(ae + ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{\overline{d}\left(\overline{\overline{a}\,\overline{\overline{d^{-1}}\,\overline{f}}} + \overline{\overline{c}\,\overline{\overline{d^{-1}}\,\overline{f}}}\right)}\right)$$
$$= \left(ae + ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{\overline{d}\,\overline{\overline{a}\,\overline{\overline{d^{-1}}\,\overline{f}}}} + \overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{d^{-1}}\,\overline{f}}}}\right)$$
$$= \left(ae + ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{a}\,\overline{\overline{d}\;\overline{d^{-1}}\;\overline{f}} + \overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{d^{-1}}\,\overline{f}}}}\right)$$
$$= \left(ae + ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{a}f + \overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{d^{-1}}\,\overline{f}}}}\right)$$
$$= (ae, \overline{a}f) + \left(ce + D\overline{d\overline{f}}, \overline{\overline{d}\,\overline{e}} + \overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{d^{-1}}\,\overline{f}}}}\right) = (a,0)(e,f) + (c,d)(e,f).$$

Now we show that $K \times^D K$ is left alternative.
Case 1: $b = 0$. Then, by Proposition 3(c), we get that

$$(a,b)^2(c,d) = (a,0)^2(c,d) = (a^2,0)(c,d) = (a^2c, \overline{a^2}d) = (a^2c, \overline{a}^2 d) =$$
$$= (a(ac), \overline{a}(\overline{a}d)) = (a,0)(ac, \overline{a}d) = (a,0)((a,0)(c,d)) = (a,b)((a,b)(c,d))$$

Case 2: $b \neq 0$ and $t(a) \neq 0$. Then, by Proposition 3(c), we get that

$$(a,b)^2(c,d) = \left(a^2 + Dn(b), \overline{\overline{b}\,\overline{a}} + \overline{\overline{b}\,\overline{a}\,\overline{\overline{\overline{b^{-1}}\,\overline{b}}}}\right)(c,d) =$$
$$= \left(a^2 + Dn(b), \overline{\overline{b}\,\overline{a}} + \overline{\overline{b}a}\right)(c,d) = \left(a^2 + Dn(b), \overline{\overline{b}\,\overline{a} + \overline{b}a}\right)(c,d) =$$
$$= \left(a^2 + Dn(b), \overline{\overline{b}(\overline{a} + a)}\right)(c,d) = (a^2 + Dn(b), t(a)b)(c,d) =$$
$$= \left((a^2 + Dn(b))c + D\overline{bt(a)\overline{d}}, \overline{\overline{b}t(a)\overline{c}} + \overline{\overline{b}t(a)\overline{\overline{(a^2 + Dn(b))}\,\overline{\overline{b^{-1}t(a)^{-1}\overline{d}}}}}\right)$$
$$= \left(a^2c + Dn(b)c + Dt(a)\overline{b\overline{d}}, t(a)\overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\left(\overline{\overline{a}^2\overline{\overline{b^{-1}}\,\overline{d}}} + Dn(b)\overline{\overline{b^{-1}}\,\overline{d}}\right)}\right)$$
$$= \left(a^2c + Dn(b)c + Dt(a)\overline{b\overline{d}}, t(a)\overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{a}^2\overline{\overline{b^{-1}}\,\overline{d}}} + Dn(b)d\right)$$

and, by Proposition 3(a), we get that

$$(a,b)((a,b)(c,d)) = (a,b)(ac + D\overline{b\overline{d}}, \overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}}) =$$
$$= \left(a(ac + D\overline{b\overline{d}} + D\overline{b(\overline{b}\overline{c} + \overline{\overline{b}\,\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}})}, \overline{\overline{b}(\overline{ac} + Db\overline{d})} + \overline{\overline{\overline{b}\,\overline{a}\,\overline{b^{-1}}(\overline{b}\overline{c} + \overline{\overline{b}\,\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}})}}\right)$$
$$= \left(a^2c + Da\overline{b\overline{d}} + Dn(b)c + Dn(b)\overline{\overline{a}\,\overline{b^{-1}}\,\overline{d}}, \overline{\overline{b}ac} + Dn(b)d + \overline{\overline{b}\,\overline{a}c} + \overline{\overline{b}\,\overline{a}^2\overline{\overline{b^{-1}}\,\overline{d}}}\right)$$
$$= \left(a^2c + Dn(b)c + Dt(a)\overline{b\overline{d}}, t(a)\overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{a}^2\overline{\overline{b^{-1}}\,\overline{d}}} + Dn(b)d\right).$$

Case 3: $b \neq 0$ and $t(a) = 0$. Then, by Proposition 3(c) and the beginning of the calculation in Case 2, we get that

$$(a,b)^2(c,d) = (a^2 + Dn(b), 0)(c,d) = \left((a^2 + Dn(b))c, \overline{(a^2 + Dn(b))}d\right) =$$
$$= \left(a^2c + Dn(b)c, \overline{a}^2 d + Dn(b)d\right).$$

By the second part of the calculation in Case 2 and the fact that $\overline{a}^2 = -n(a) \in k$, we get that

$$(a,b)((a,b)(c,d)) = \left(a^2c + Dn(b)c, \overline{\overline{b}\,\overline{a}^2\overline{\overline{b^{-1}}\,\overline{d}}} + Dn(b)d\right) =$$
$$= \left(a^2c + Dn(b)c, \overline{a}^2 d + Dn(b)d\right).$$

The last part follows from the fact that $\overline{(a,b)}(a,b) = (n(a) - Dn(b), 0)$. $\square$

Corollary 4. *Suppose that k is an ordered field equipped with the identity conjugation. If we for each nonnegative integer i choose a negative $D_i \in k$ and recursively define the k-algebras k^i, for $i \geq 0$, by $k^0 = k$ and $k^i = k^{i-1} \times^{D_i} k^{i-1}$, for $i \geq 1$, then, for all $i \geq 1$, an $a \in k^i$ satisfies $n(a) = 1$ if and only if there is a nonzero $b \in k^i$ satisfying $\overline{b}a = b$.*

Proof. This follows immediately from Theorem 1 and Proposition 9. □

Remark 5. Now we use our version of Hilbert 90 to parametrize the solutions to the norm one equation for 2^n-ons. Suppose that k is an ordered field and that the k-algebras k^i, for $i \geq 0$, are defined as in Corollary 4. Fix a nonnegative integer n. Now we define 2^n coordinates on k^n recursively in the following way. Suppose that $x = (y, z) \in k^n = k^{n-1} \times^{D_n} k^{n-1}$ and $1 \leq j \leq 2^n$. The jth coordinate of x is defined as the jth coordinate of y, if $1 \leq j \leq 2^{n-1}$, and as the jth coordinate of z, if $2^{n-1} < j \leq 2^n$. Label the 2^n coordinates of x as $x_1, x_2, \ldots, x_{2^n}$. Put $C_i = -D_i$, for $i \geq 0$. Define a total order $\prec$ on the 2^n elements of $\{0,1\}^n$ by letting $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$ if there is a positive integer j with $a_j = 0$, $b_j = 1$ and $a_i = b_i$ for $i > j$. Define $e(1), e(2), \ldots, e(2^n) \in \{0,1\}^n$ by $e(1) \prec e(2) \prec \cdots \prec e(2^n)$. For each i, let $C^{e(i)}$ denote the product $\prod_{j=1}^n C_j^{e(i)_j}$, where $e(i)_j$ denotes the jth coordinate of $e(i)$, for $j = 1, \ldots, n$. Then, by equation (2), we get that

$$n(x) = \sum_{i=1}^{2^n} C^{e(i)} x_i^2 = 1$$

if and only if there are $s_j \in k$, for $1 \leq j \leq 2^n$, not all zero, such that

$$x_1 = \frac{s_1^2 - \sum_{j=2}^{2^n} C^{e(j)} s_j^2}{\sum_{j=1}^{2^n} C^{e(j)} s_j^2}$$

and

$$x_i = \frac{2 s_1 s_i}{\sum_{j=1}^{2^n} C^{e(j)} s_j^2}$$

for $2 \leq i \leq 2^n$.

A special case of the above discussion, interesting on it's own right, is the following. A collection of elements $x_1, x_2, \ldots, x_{2^n}$ of an ordered field k satisfy

$$x_1^2 + x_2^2 + \cdots x_{2^n}^2 = 1$$

if and only if there are $s_j \in k$, for $1 \leq j \leq 2^n$, not all zero, such that

$$x_1 = \frac{s_1^2 - (s_2^2 + s_3^2 + \cdots s_{2^n}^2)}{s_1^2 + s_2^2 + \cdots s_{2^n}^2}$$

and

$$x_i = \frac{2 s_1 s_i}{s_1^2 + s_2^2 + \cdots s_{2^n}^2}$$

for $2 \leq i \leq 2^n$. In particular, this implies that a vector

$$x = (x_1, x_2, \ldots, x_{2^n+1})$$

of 2^n+1 integers satisfy

$$x_1^2 + x_2^2 + \cdots + x_{2^n}^2 = x_{2^n+1}^2$$

if and only if x is a rational multiple of a vector of the form

$$\left(s_1^2 - (s_2^2 + s_3^2 + \cdots s_{2^n}^2), 2s_1s_2, 2s_1s_3, \ldots, 2s_1s_{2^n}, s_1^2 + s_2^2 + \cdots + s_{2^n}^2\right)$$

for some integers $s_1, s_2, \ldots, s_{2^n}$. Note that this result is stated without proof on p. 72 in [1].

We end this section with a digression on norm multiplicativity of Conway-Smith doubles.

Lemma 1. *If $T : K \to K$ is a k-linear map satisfying the following three conditions (i) $T(ab) = T(ba)$; (ii) $T(\overline{a}) = T(a)$; (iii) $T\left(\overline{a(bc)}\right) = T\left(\overline{c}\left(\overline{b}\overline{a}\right)\right)$, for all $a, b, c \in K$, then*

$$T\left(\overline{a\overline{b\overline{c\overline{d\overline{e}}}}}\right) = T\left(\overline{\overline{e}\,\overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{b}\overline{a}}}}}\right)$$

for all $a, b, c, d, e \in K$.

Proof. We iterate (iii) with the aid of (i) and (ii). Take $a, b, c, d, e \in K$. Then

$$T\left(\overline{a\overline{b\overline{c\overline{d\overline{e}}}}}\right) = T\left(\overline{a}\left(b\overline{c\overline{d\overline{e}}}\right)\right) = T\left(\left(c\overline{d\overline{e}}\right)\overline{b}\overline{a}\right) =$$

$$= T\left(d\overline{e}\left(\overline{c}\,\overline{\overline{b}\overline{a}}\right)\right) = T\left(e\left(\overline{d}\,\overline{\overline{c}\,\overline{\overline{b}\overline{a}}}\right)\right) = T\left(\overline{\overline{e}\,\overline{\overline{d}\,\overline{\overline{c}\,\overline{\overline{b}\overline{a}}}}}\right).$$

□

Proposition 10. *Suppose that K is left alternative, left distributive, weak right distributive and has k-valued trace and anisotropic k-valued norm. Suppose also that there is a k-linear map $T : K \to K$ satisfying the following three conditions (i) $T(ab) = T(ba)$; (ii) $T(\overline{a}) = T(a)$; (iii) $T\left(\overline{a(bc)}\right) = T\left(\overline{c}\left(\overline{b}\overline{a}\right)\right)$ for all $a, b, c \in K$. If we extend the map T to $K \times^D K$ by the relation $T((a, b)) = T(a)$, for $a, b \in K$, then (i), (ii) and (iii) hold for all $a, b, c \in K \times^D K$. In that case, the norm on $K \times^D K$ is multiplicative.*

Proof. Note that by Propositions 2 and 9, the assumptions on K imply that K is a division algebra with $a^{-1} = n(a)^{-1}\overline{a}$, for all nonzero $a \in K$. It is straightforward to show (i) and (ii). Now we show (iii). Take $a, b, c, d, e, f \in K$ and put $x = (a, b)$, $y = (c, d)$ and $z = (e, f)$. We have to check five different cases.

Case 1: at least two of b, d and f are equal to zero. Then, by (iii), we get that $T\left(\overline{x(yz)}\right) = T\left(\overline{a(ce)}\right) = T\left(\overline{e}(\overline{c}\;\overline{a})\right) = T\left(\overline{z}\left(\overline{y}\;\overline{x}\right)\right)$.

Case 2: $b=0$, $d\neq 0$ and $f\neq 0$. Then, by left distributivity and (iii), we get that

$$T\left(\overline{x(yz)}\right)=T\left(\overline{(a,0)((c,d)(e,f))}\right)=T\left(\overline{a\left(ce+D\overline{d\overline{f}}\right)}\right)=$$

$$=T\left(\overline{a(ce)}\right)+DT\left(\overline{a\overline{d\overline{f}}}\right)=T\left(\overline{e}(\overline{c}\;\overline{a})\right)+DT\left(\overline{f(\overline{d}a)}\right)=$$

$$=T\left(\overline{e}(\overline{c}\;\overline{a})+D\overline{f(\overline{d}a)}\right)=T\left((\overline{e},-f)(\overline{c}\;\overline{a},\overline{(\overline{-d})a})\right)=T\left(\overline{z}(\overline{y}\;\overline{x})\right).$$

Case 3: $b\neq 0$, $d=0$ and $f\neq 0$. Then, by left distributivity and (iii), we get that

$$T\left(\overline{x(yz)}\right)=T\left(\overline{(a,b)((c,0)(e,f))}\right)=T\left(\overline{(a,b)(ce,cf)}\right)=$$

$$=T\left(\overline{a(ce)}\right)+DT\left(b\overline{cf}\right)=T\left(\overline{e}(\overline{c}\;\overline{a})\right)+DT\left(f\overline{c}\overline{b}\right)=T\left(\overline{e}(\overline{c}\;\overline{a})+Df\overline{c}\overline{b}\right)=$$

$$=T\left((\overline{e},-f)(\overline{c}\;\overline{a},-\overline{c}b)\right)=T\left(\overline{(e,f)}\left(\overline{(c,0)}\;\overline{(a,b)}\right)\right)=T\left(\overline{z}(\overline{y}\;\overline{x})\right).$$

Case 4: $b\neq 0$, $d\neq 0$ and $f=0$. Then, by left distributivity and (iii), we get that

$$T\left(\overline{x(yz)}\right)=T\left(\overline{(a,b)((c,d)(e,0))}\right)=T\left(\overline{(a,b)\left(ce,\overline{\overline{d}\,\overline{e}}\right)}\right)=$$

$$=T\left(\overline{a(ce)}\right)+DT\left(\overline{b(\overline{\overline{d}\,\overline{e}})}\right)=T\left(\overline{e}(\overline{c}\;\overline{a})\right)+DT\left(\overline{e}\,\overline{d\overline{b}}\right)=$$

$$=T\left(\overline{e}(\overline{c}\;\overline{a})+D\overline{e}\,\overline{d\overline{b}}\right)=T\left(\overline{e}\left(\overline{c}\;\overline{a}+D\overline{d\overline{b}}\right)\right)=T\left(\overline{z}(\overline{y}\;\overline{x})\right).$$

Case 5: $b\neq 0$, $d\neq 0$ and $f\neq 0$. Then, by left distributivity, Lemma 1 and (iii), we get that

$$T\left(\overline{x(yz)}\right)=T\left(\overline{(a,b)\left(ce+D\overline{d\overline{f}},\overline{\overline{d}\,\overline{e}}+\overline{\overline{d}\,\overline{\overline{\overline{\overline{c}d^{-1}}\;\overline{f}}}}\right)}\right)=$$

$$=T\left(\overline{a(ce+D\overline{d\overline{f}})+Db\left(\overline{d}\,\overline{e}+\overline{\overline{d}\,\overline{\overline{\overline{\overline{c}d^{-1}}\;\overline{f}}}}\right)}\right)$$

$$=T\left(\overline{a(ce)}\right)+DT\left(a\left(\overline{d\overline{f}}\right)\right)+DT\left(b(\overline{d}\,\overline{e})\right)+DT\left(b\left(\overline{d}\,\overline{\overline{\overline{\overline{c}d^{-1}}\;\overline{f}}}\right)\right)$$

$$=T\left(\overline{e}(\overline{c}\;\overline{a})+DT\left(f(\overline{d}a)\right)+DT\left(\overline{e}\,\overline{d\overline{b}}\right)+T\left(f\left(\overline{d}\,\overline{\overline{\overline{cd^{-1}}\overline{b}}}\right)\right)\right.$$

$$=T\left(\overline{e}(\overline{c}\;\overline{a}+D\overline{d\overline{b}})+Df\left(\overline{d}\,\overline{\overline{\overline{cd^{-1}}\overline{b}}}\right)\right)$$

$$=T\left((\overline{e},-f)\left(\overline{c}\;\overline{a}+D\overline{d\overline{b}},-\overline{\overline{d}a}-\overline{\overline{d}\,\overline{\overline{\overline{cd^{-1}}\overline{b}}}}\right)\right)=T\left(\overline{z}(\overline{yx})\right).$$

The last part follows from the above and Proposition 4. $\square$

Corollary 5. *Every k-algebra k^i, for $i\geq 0$, as defined in Corollary 3, has multiplicative norm.*

Proof. Suppose that we define the k-linear maps $T_i : k^i \to k$, for all $i \geq 0$, inductively by $T_0 = \mathrm{id}_k$ on $k^0 = k$ and $T_{i+1}((a, b)) = T_i(a)$, for $i \geq 0$ and $(a, b) \in k^{i+1}$. Since obviously id_k satisfies (i), (ii) and (iii) from Proposition 10, it follows that the same is true for T_i, for any $i \geq 0$. Therefore, by Propositions 4 and 9, the norm is multiplicative on k^i, for any $i \geq 0$. □

References

[1] T. Andreescu and D. Andrica, *An Introduction to Diophantine Equations*, GIL Publishing House (2002).
[2] N. Bourbaki, *Algebra I*, Chapters 1-3, Springer (2008).
[3] J. H. Conway and Derek Smith, *On quaternions and octonions*, A. K. Peters (2003).
[4] D. S. Dummit och R. M. Foote, Abstract Algebra, Wiley (2003).
[5] R. Dwilewicz, J. Minac, A. Schultz and J. Swallow, Hilbert 90 for biquadratic extensions, *Amer. Math. Monthly* **114**, 577–587 (2007).
[6] N. D. Elkies, Pythagorean triples and Hilbert's Theorem 90, available at <http://www.math.harvard.edu/∼elkies/>.
[7] D. Hilbert, *The theory of algebraic number fields*, Springer-Verlag, Berlin (1998).
[8] A. Hurwitz, Über die Komposition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen* 309–316 (1898).
[9] S. Lang, *Algebra*, Addison-Wesley (1993).
[10] F. Lorenz, Ein Scholion zum Satz 90 von Hilbert, *Abh. Math. Sem. Univ. Hamburg* **68**, 347–362 (1998).
[11] T. Ono, Variations on a Theme of Euler: Quadratic Forms, Elliptic Curves and Hopf Maps, Springer (1994).
[12] A. Pfister, Zur Darstellung definiter Funktionen als Summe von Quadraten, *Inventiones Mathematicae* **4**, 229–237 (1967).
[13] A. Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, London Math. Lect. Notes 217, Cambridge Univ. Press (1995).
[14] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Dover Publications (1994).
[15] J. D. H. Smith, A Left Loop on the 15-Sphere, *Journal of Algebra* **176**, 128–138 (1995).
[16] W. D. Smith, Quaternions, octonions, and now, 16-ons and 2^n-ons; New kinds of numbers (2004). Available at http://www.math.temple.edu/∼wds/homepage/works.html
[17] A. Speiser, Zahlentheoretische Sätze aus der Gruppentheorie, *Math. Z.* **5**, 1–6 (1919).
[18] O. Taussky, Sums of Squares, Amer. Math. Monthly, Vol. 77, 805-830 (1970).

Patrik Lundström, University West, Department of Engineering Science, Trollhättan, Sweden

E-mail address: Patrik.Lundstrom@hv.se