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# FUZZIFIED CATEGORIES OF COMPOSITION GRAPHS

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Abstract. Serving as a generalization of many examples of fuzzy algebraical systems equipped with a binary operation, we introduce fuzzy composition graphs and show that categories formed by such graphs are, in the sense of Wyler [10], top categories. By using this, we investigate projective and injective objects in such categories, and we determine when various limits and colimits, such as terminal and initial objects, products, coproducts, pullbacks, pushouts, equalizers, coequalizers, kernels and cokernels, exist in categories of this type and what they look like. These results are then applied to the categories of fuzzy sets, fuzzy categories, fuzzy groupoids, fuzzy monoids, fuzzy groups and fuzzy abelian groups.

## 1. Introduction

Almost forty years ago, Zadeh [11] introduced the notion of a fuzzy set as a function from the given set to the unit interval. Six years later, Rosenfeld [9] extended the concept of fuzzy sets to algebra by defing fuzzy subgroups of a group. Since then a lot of work has been devoted to proposing different versions of what fuzzy algebras of distinct types may be. In particular, Negoita and Ralesco [6] introduced the notion of fuzzy modules, Pan [7] constructed the category $R$-fzmod of fuzzy left modules over an associative ring $R$ and Permouth [8] defined the tensor product of fuzzy modules. In [4] López-Permouth and Malik further categorized the study of fuzzy modules by showing that $R$-fzmod is a top category, in the sense of Wyler [10], over $R$-mod, the category of left modules, induced by a certain topological theory on $R$-mod. In the same article, López-Permouth and Malik [4] make the important observation that, by general results of Wyler [10] concerning universal functors in top categories, properties of $R$-mod, such as, completeness and cocompleteness, are inherited by the top category $R$-fzmod.

In this article, we adapt this observation to cover other fuzzified versions of algebraical systems such as sets, monoids, groupoids and groups. These algebraical systems are categories or, even more generally, composition graphs. In fact, sets are discrete categories, monoids are categories with with one object, groupoids are categories where all arrows are isomorphisms and groups are monoids that are groupoids. Our main objective is thus to show that fuzzified categories of composition graphs are top categories. To this end, we, in Section 2, recall the definition and some results concerning top categories by Wyler [10]. There, we show, for the convenience of the reader, the existence of limits and colimits in top categories as a specialization of a result by Wyler [10] concerning universal functors in top categories. In this section, we also analyze monics, epics, isomorphisms, and projective and injective objects in top categories. In Section 3, we recall briefly some of the ideas and results by López-Permouth and Malik [4] concerning fuzzy modules. The results in this section are not used in the sequel, but will instead serve as a motivation for the approach taken later. In Section 4, we introduce fuzzy composition graphs. There, we show, among other things, that categories of fuzzy composition graphs are top categories. This makes it possible for us, in Section 5, to use Wylers results concerning top categories for, in particular, the existence of various types of limits and colimits, such as, initial and terminal objects, products and coproducts, pullbacks, pushouts, equalizers, coequalizers, kernels and cokernels. Also abelian categories are analyzed. In Section 6, we exemplify the results from Section 5 for the categories of fuzzy sets, fuzzy categories, fuzzy groupoids, fuzzy monoids, fuzzy groups and fuzzy abelian groups.

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## 2. Composition graphs

We will use the following conventions on composition graphs and categories. A graph $\mathbf{A}$ is a family of objects $\mathrm{ob}(\mathbf{A})$ and and a family of arrows $\mathrm{ar}(\mathbf{A})$ and two operations, domain and codomain, that assigns to every arrow $\alpha$ in $\mathbf{A}$ the objects $\mathrm{d}(\alpha)$ and $\mathrm{c}(\alpha)$ respectively. We will indicate this by writing $\alpha : \mathrm{d}(\alpha) \to \mathrm{c}(\alpha)$. The collection of arrows from $a$ to $b$ in $\mathbf{A}$ is denoted $\mathrm{Hom}_{\mathbf{A}}(a,b)$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are graphs. A covariant (or contravariant) functor $f : \mathbf{A} \to \mathbf{B}$ is a pair of maps $f_{\mathrm{ob}} : \mathrm{ob}(\mathbf{A}) \to \mathrm{ob}(\mathbf{B})$ and $f_{\mathrm{ar}} : \mathrm{ar}(\mathbf{A}) \to \mathrm{ar}(\mathbf{B})$ satisfying $f_{\mathrm{ar}}(\alpha) : f_{\mathrm{ob}}(a) \to f_{\mathrm{ob}}(b)$ (or $f_{\mathrm{ar}}(\alpha) : f_{\mathrm{ob}}(b) \to f_{\mathrm{ob}}(a)$) for all arrows $\alpha : a \to b$ in $\mathbf{A}$. A composition graph is a graph $\mathbf{A}$ with a rule that to each pair of arrows $\alpha : a \to b$ and $\beta : b \to c$ in $\mathbf{A}$ associates an arrow $\beta\alpha : a \to c$ in $\mathbf{A}$. A composition graph is called a category if there are identity arrows $\mathrm{id}_a$ at each object $a$ of the graph and the composition is associative whenever it is defined. A composition graph is called small if the family of arrows is a set. A functor of composition graphs (or categories) is a functor of the underlying graphs respecting the composition (and the identity elements). We will use the following notation.

- **Set** The category with small sets as objects and set mappings as arrows.
- **Cat** The category with small categories as objects and functors as arrows.
- **Grd** The category with small groupoids as objects and groupoid homomorphisms as arrows.
- **Mon** The category with small monoids as objects and monoid homomorphisms as arrows.

**Grp** The category with groups as objects and group homomorphisms as arrows.

**Ab** The category with abelian groups as objects and group homomorphisms as arrows.

**Cord** The category of complete ordered lattices and order preserving mappings. Moreover, infima and suprema in **Cord** are denoted by $\wedge$ and $\vee$ respectively. We say that an arrow $f : X \to Y$ in **Cord** preserves infima (or suprema) if $f(\wedge_{i\in I} x_i) = \wedge_{i\in I} f(x_i)$ (or $f(\vee_{i\in I} x_i) = \vee_{i\in I} f(x_i)$) for every family $\{x_i\}_{i\in I}$ of elements of $X$ including the empty family.

## 3. Top categories

In this section, we first recall Wyler's [10] definition of top categories (see Definition 1). Then we investigate monics, epics and isomorphisms in top categories (see Proposition 1). This is in turn used to investigate projective and injective objects in top categories (see Proposition 2). Thereafter, we, for the convenience of the reader, show the existence of limits and colimits in top categories (see Proposition 3) as a specialization of a result by Wyler [10] concerning universal functors in top categories. In the end of this section, we determine when top categories are abelian (see Proposition 4).

**Definition 1.** Let $\mathbf{C}$ denote a category and suppose that we have a contravariant functor $s : \mathbf{C} \to \mathbf{Cord}$ with the property that $s(\alpha)$ preserves infinima for all arrows $\alpha$ in $\mathbf{C}$. Then the top category $\mathbf{C}^s$ over $\mathbf{C}$ is defined as follows. Objects of $\mathbf{C}^s$ are pairs $(a, x)$ with $a \in \mathrm{ob}(\mathbf{C})$ and $x \in s(a)$. An arrow $(a, x) \to (b, y)$ in $\mathbf{C}^s$ is a triple $(x, \alpha, y)$ where $\alpha : a \to b$ is an arrow in $\mathbf{C}$ and $x \in s(a)$, $y \in s(b)$, and $x \le s(\alpha)(y)$ in $s(a)$. Composition in $\mathbf{C}^s$ is defined by putting $(y, \beta, z)(x, \alpha, y) = (x, \beta\alpha, z)$ whenever $\beta\alpha$ is defined in $\mathbf{C}$. The identity morphisms are given by $\mathrm{id}_{(a,x)} = (x, \mathrm{id}_a, x)$.

**Remark 1.** (a) Top categories are fibred in the sense of Grothendieck [3].

(b) Given a top category $\mathbf{C}^s$ over $\mathbf{C}$, there is, by Lemma 4.1 in [10], a canonical choice of a covariant functor $t : \mathbf{C} \to \mathbf{Cord}$ with the property that $t(\alpha)$ respects suprema for all arrows $\alpha$ in $\mathbf{C}$. In fact, the functor $t$ is defined by $t(a) = s(a)$ for all objects $a$ in $\mathbf{C}$ and on arrows by the condition that $t(\alpha)(x) \le y$ holds precisely when $x \le s(\alpha)(y)$ holds for all arrows $\alpha : a \to b$ in $\mathbf{C}$ and all $x \in s(a)$, $y \in s(b)$. Note that the last condition can be reformulated by saying that $t(\alpha)(x) = \inf\{y \in Y \mid x \le s(\alpha)(y)\}$.

Assume for the rest of the section that $s : \mathbf{C} \to \mathbf{Cord}$ is a contravariant functor of categories. Now we examine some basic properties of top categories. To do that, we need another definition from Wyler [10] and a lemma.

**Definition 2.** Let $F : \mathbf{I} \to \mathbf{C}$ be a functor. We say that a functor $G : \mathbf{I} \to \mathbf{C}^{\mathrm{top}}$ lifts $F$ if $PG = F$, where $P : \mathbf{C}^{\mathrm{top}} \to \mathbf{C}$ is the functor defined by $P(a, x) = a$ for all objects $a$ in $\mathbf{C}$ and $P(x, \alpha, y) = \alpha$ for all arrows $\alpha : a \to b$

in $\mathbf{C}$ and all $x \in s(a)$ and all $y \in s(b)$ such that $x \leq s(\alpha)(y)$. Note that we will often write $\underline{G}$ instead of $PG$.

**Lemma 1.** *Let $F : \mathbf{I} \to \mathbf{C}$ be a functor and suppose that $x \in s(F(i))$. If the object $i$ in $\mathbf{I}$ has the property that there are no arrows $\alpha$ in $\mathbf{I}$ with $\mathrm{d}(\alpha) = i$ (or $\mathrm{c}(\alpha) = i$) and $F(\alpha)$ non identity, then $F$ can be lifted to a functor $G : \mathbf{I} \to \mathbf{C}^{\mathrm{top}}$ with $G(i) = (F(i), x)$.*

*Proof.* Define the functor $G : \mathbf{I} \to \mathbf{C}^{\mathrm{top}}$ on objects by $G(i) = (F(i), x)$ and $G(j) = (F(j), x_j)$, for all $j \in \mathrm{ob}(\mathbf{I}) \setminus \{i\}$, where $x_j$ is the smallest (or greatest) element in $s(F(j))$, and on arrows $\alpha : j \to k$ in $\mathbf{I}$ by $G(\alpha) = (x_j, F(\alpha), x_k)$. □

Recall that an arrow $\alpha$ in a category $\mathbf{C}$ is called monic (or epic) when any equality $\alpha\beta = \alpha\gamma$ (or $\beta\alpha = \gamma\alpha$) implies that $\beta = \gamma$. An arrow is called an isomorphism if it has a two sided inverse.

**Proposition 1.** *Let $(x, \alpha, y) : (a, x) \to (b, y)$ be an arrow in $\mathbf{C}^{\mathrm{top}}$.*

(a) *$(x, \alpha, y)$ is monic if and only if $\alpha$ is monic.*
(b) *$(x, \alpha, y)$ is epic if and only if $\alpha$ is epic.*
(c) *$(x, \alpha, y)$ is an isomorphism if and only if $\alpha$ is an isomorphism.*

*Proof.* (a) If $\alpha$ is monic, then obviously $(x, \alpha, y)$ is monic. On the other hand, suppose that $(x, \alpha, y)$ is monic. Choose arrows $\beta : c \to a$ and $\gamma : c \to a$ in $\mathbf{C}$ with $\alpha\beta = \alpha\gamma$. By Lemma 1, there is $z \in s(c)$ such that $(z, \beta, x)$ and $(z, \gamma, x)$ are arrows in $\mathbf{C}^{\mathrm{top}}$ and $(x, \alpha, y)(z, \beta, x) = (x, \alpha, y)(z, \gamma, x)$. Since $(x, \alpha, y)$ is monic, we get that $(z, \beta, x) = (z, \gamma, x)$ which in turn implies that $\beta = \gamma$. (b) is proved in the same way as (a). (c) is trivial. □

Recall that an object $a$ in a category $\mathbf{C}$ is called projective (or injective) if for all arrows $\alpha : a \to b$ (or $\alpha : b \to a$) and all epic $\beta : c \to b$ (or monic $\beta : b \to c$), there is $\gamma : a \to c$ (or $\gamma : c \to a$) such that $\alpha = \beta\gamma$ (or $\alpha = \gamma\beta$).

**Proposition 2.** *An object $(a, x)$ in $\mathbf{C}^{\mathrm{top}}$ is projective (or injective) if and only if $a$ is projective (or injective) in $\mathbf{C}$ and $x$ is the least (or greatest) element in $s(a)$.*

*Proof.* We show the projective part of the proof. The injective part is proved in the same way. Suppose that $(a, x)$ is projective. By Lemma 1 and Proposition 1(b), we get that $a$ is projective. Furthermore, let $x_0$ and $x_1$ denote respectively the least and greatest element in $s(a)$. If $x > x_0$, then there is no arrow $(x_0, \mathrm{id}_a, x) : (a, x_0) \to (a, x)$ although there is an epic arrow $(x, \mathrm{id}_a, x_1) : (a, x) \to (a, x_1)$ and an arrow $(x_0, \mathrm{id}_a, x_1) : (a, x_0) \to (a, x_1)$. This contradicts the fact that $(a, x)$ is projective. On the other hand, if $a$ is projective, then, trivially, $(a, x)$ is projective. □

Let $\mathbf{I}$ be a small category. Recall that a limit of a functor $F : \mathbf{I} \to \mathbf{C}$ is an object $\varprojlim F_i$ in $\mathbf{C}$ and a family of arrows $f_i : \varprojlim F(i) \to F(i)$, $i \in \mathrm{ob}(\mathbf{I})$, satisfying $f_j = F(\alpha) f_i$ for all arrows $\alpha : i \to j$ in $\mathbf{I}$, with the following universal property. For every object $X$ in $\mathbf{C}$ and every family of arrows $g_i : X \to F(i)$ with $g_j = F(\alpha)g_i$ for all arrows $\alpha : i \to j$ in $\mathbf{I}$, there is a unique arrow $f : X \to \varprojlim F_i$ satisfying $g_i = f_i f$ for all $i \in \text{ob}(\mathbf{I})$. A colimit $\varinjlim F_i$ of $F$ is a limit of the functor $F^{\text{op}} : \mathbf{I} \to \mathbf{C}^{\text{op}}$. A category is called complete (cocomplete) if all limits (colimits) for a small category $\mathbf{I}$ exist in the category. Let $\mathbf{S}$ be a subcategory of $\mathbf{I}$. Assume that a limit $\varprojlim F_i$ of $F$ exists. We say that $\mathbf{S}$ is sufficient for $\varprojlim F_i$ if for all $i \in \text{ob}(\mathbf{S})$ there is $j \in \text{ob}(\mathbf{I})$ and an arrow $\alpha : i \to j$ such that $f_j = F(\alpha)f_i$. If a colimit $\varinjlim F_i$ of $F$ exists, then we say that $\mathbf{S}$ is cosufficient for $F$ if it is sufficient for $F^{\text{op}}$.

**Proposition 3.** *Let* $\mathbf{I}$ *be small category. Assume that* $F : \mathbf{I} \to \mathbf{C}^{\text{top}}$ *is a functor with* $F(i) = (F_i, x_i)$, $i \in \text{ob}(\mathbf{I})$, *for some* $F_i \in \text{ob}(\mathbf{C})$, *and some* $x_i \in s(F_i)$.

(a) *A limit of* $F$ *exists if and only if a limit of* $\underline{F}$ *exists. In that case a limit of* $F$ *is given by* $(\varprojlim F_i, \wedge_{i \in \text{ob}(\mathbf{S})} s(f_i)(x_i))$ *where* $f_i : \varprojlim F_i \to F_i$ *are the arrows associated with the choice of limit of* $\underline{F}$ *and* $\mathbf{S}$ *is any sufficient subcategory of* $\mathbf{I}$. *In particular,* $\mathbf{C}^{\text{top}}$ *is complete if and only if* $\mathbf{C}$ *is complete.*

(b) *A colimit of* $F$ *exists if and only if a colimit of* $\underline{F}$ *exists. In that case a colimit of* $F$ *is given by* $(\varinjlim F_i, \vee_{i \in \text{ob}(\mathbf{S})} t(f'_i)(x_i))$ *where* $f'_i : F_i \to \varinjlim F_i$ *are the arrows associated with the choice of colimit of* $\underline{F}$ *and* $\mathbf{S}$ *is any cosufficient subcategory of* $\mathbf{I}$. *In particular,* $\mathbf{C}^{\text{top}}$ *is cocomplete if and only if* $\mathbf{C}$ *is cocomplete.*

*Proof.* We show (a). The proof of (b) is dual. It is enough to prove the result for $\mathbf{S} = \mathbf{I}$. In fact, if $i \in \mathbf{S}$, $j \in \mathbf{I}$ and $\alpha : i \to j$, then, by the equality $f_j = \underline{F}(\alpha)f_i$, we get that $s(f_j)(x_j) = s(\underline{F}(\alpha)f_i)(x_j) = s(f_i)s(\underline{F}(\alpha))(x_j) \geq s(f_i)(x_i)$.

Suppose that $(a, x)$ is a limit of $F$ for some $a \in \text{ob}(\mathbf{C})$ and $x \in s(a)$ with corresponding arrows $\overline{\alpha}_i := (x, \alpha_i, x_i) : (a, x) \to (F_i, x_i)$, $i \in \text{ob}(\mathbf{I})$. Then $a$ is a limit of $\underline{F}$. In fact, choose an object $b$ in $\mathbf{C}$ and arrows $g_i : b \to F_i$, $i \in \text{ob}(\mathbf{I})$ such that $g_i = \underline{F}(\alpha)g_j$ for all arrows $\alpha : i \to j$ in $\mathbf{I}$. If we let $y$ denote the least element in $s(a)$, then there are induced arrows $\overline{g}_i : (b, y) \to (F_i, x_i)$ in $\mathbf{C}^{\text{top}}$ with the property that $\overline{g}_i = F(\alpha)\overline{g}_j$ for all arrows $\alpha : i \to j$ in $\mathbf{I}$. Then there is a (unique) arrow $h : (b, y) \to (a, x)$ such that $\overline{g}_i = \overline{\alpha}_i h$ for all $i \in \text{ob}(\mathbf{I})$. Hence, there is an induced arrow $\underline{h} : b \to a$ such that $g_i = \alpha_i \underline{h}$ for all $i \in \text{ob}(\mathbf{I})$. Suppose that we are given another arrow $h' : b \to a$ in $\mathbf{C}$ with the property that $g_i = \alpha_i h'$. Then $h'$ lifts to an arrow $\overline{h'} : (b, y) \to (a, x)$ such that $\overline{g}_i = \overline{\alpha}_i \overline{h'}$ for all $i \in \text{ob}(\mathbf{I})$. By the uniqueness of $h$ we get that $\underline{h} = h'$.

Suppose now that $a \in \text{ob}(\mathbf{C})$ is a limit of $\underline{F}$. Put $x = \wedge_{i \in \text{ob}\mathbf{I}} s(f_i)(x_i)$. We show that $(a, x)$ is a limit of $F$. By the definition of $x$, we get that $\overline{f}_i := (x, f_i, x_i) : (a, x) \to (F_i, x_i)$ is an arrow in $\mathbf{C}^{\text{top}}$ for all $i \in \text{ob}(\mathbf{I})$ and that $\overline{f}_i = F(\alpha)\overline{f}_j$ for all arrows $\alpha : i \to j$ in $\mathbf{I}$. Suppose that we are given $(b, y)$ in $\mathbf{C}^{\text{top}}$ and arrows $\overline{g}_i := (y, g_i, x_i) : (b, y) \to (F_i, x_i)$, $i \in \text{ob}(\mathbf{I})$, such that $g_i = F(\alpha)g_j$ for all $\alpha : i \to j$ in $\mathbf{I}$. Then there is an induced arrow $f : b \to a$ in $\mathbf{C}$ with the property that $g_i = f_i f$ for all $i \in \mathrm{ob}\mathbf{I}$. We need to show that $\overline{f} := (y, f, x) : (b, y) \to (a, x)$ is an arrow in $\mathbf{C}^s$. Since $s(f)$ respects infima and $s(g_i)(x_i) \geq y$ for all $i \in \mathrm{ob}\mathbf{I}$, we get that $s(f)(x) = \wedge_{i\in\mathbf{I}} s(f)s(f_i)(x_i) = \wedge_{i\in\mathbf{I}} s(f_i f)(x_i) = \wedge_{i\in\mathbf{I}} s(g_i)(x_i) \geq y$. □

Recall that a category is an Ab-category if each Hom-set is an additive abelian group and the composition is bilinear with respect to this addition. An Ab-category is additive if it has a zero object and finite products and finite coproducts exist. An additive category is called abelian if kernels and cokernels exist and every morphism whose kernel (cokernel) is zero is the kernel (cokernel) of its cokernel (kernel).

**Proposition 4.** *Let $\mathbf{C}$ be an Ab-category. Assume that*

$$s(\alpha + \alpha')(x) \geq s(\alpha)(x) \wedge s(\alpha')(x)$$

*for all arrows $\alpha, \alpha' : a \to b$ in $\mathbf{C}$ and all $x \in s(b)$.*

(a) $\mathbf{C}^{\mathrm{top}}$ *is an Ab-category.*
(b) $\mathbf{C}^{\mathrm{top}}$ *is additive if and only if $\mathbf{C}$ is additive.*
(c) $\mathbf{C}^{\mathrm{top}}$ *is abelian if and only if $\mathbf{C}$ is abelian and $s(a)$ is a singleton set for all objects $a$ in $\mathbf{C}$.*

*Proof.* (a) For arrows $(x, \alpha, y)$ and $(x, \alpha', y)$ from $(a, x)$ to $(b, y)$, put $(x, \alpha, y) + (x, \alpha', y) = (x, \alpha + \alpha', y)$. It is easy to check that this addition makes $\mathbf{C}^{\mathrm{top}}$ an Ab-category.

(b) follows from (a) and Proposition 3.

(c) By (b) we get that $\mathbf{C}^{\mathrm{top}}$ is abelian if and only if $\mathbf{C}$ is abelian and that the equation $(*)$ $s(\alpha)(y) = x$ holds for all arrows $(x, \alpha, y) : (a, x) \to (b, y)$ in $\mathbf{C}^{\mathrm{top}}$. Assume that $(*)$ holds. Seeking a contradiction, suppose that there is an object $a$ in $\mathbf{C}$ such that $s(a)$ contains two distinct elements $x$ and $y$. We can assume that $x < y$. Then $(x, \mathrm{id}_a, y)$ is an arrow in $\mathbf{C}^{\mathrm{top}}$ with the property that $s(\mathrm{id}_a)(y) = y > x$, which contradicts $(*)$. On the other hand, if $s(a)$ is a singleton set for all $a \in \mathrm{ob}(\mathbf{C})$, then, trivially, $(*)$ holds. □

## 4. Fuzzy Modules

In this section, we begin by recalling the definition of fuzzy modules (see Definition 3). Then, we use the results from the previous section to reprove some of the results from [4] concerning categories of fuzzy modules (see Proposition 5 and Proposition 6). The results in this section will not be used in the sequel. They will instead serve as a motivation for the approach taken later.

**Definition 3.** Let $R$ be a ring, $M$ a left $R$-module and $\mu$ a function from $M$ to $[0, 1]$. The pair $(M, \mu)$ is called a fuzzy left $R$-module if $\mu$ satisfies the following four conditions

(i) $\mu(x + y) \geq \min(\mu(x), \mu(y))$
(ii) $\mu(-x) = \mu(x)$

(iii) $\mu(0) = 1$
(iv) $\mu(rx) \geq \mu(x)$

for all $x, y \in M$ and all $r \in R$. An arrow of fuzzy modules $(M, \mu) \to (N, \nu)$ is an $R$-module homomorphism $f : M \to N$ satisfying $\mu(x) \leq \nu(f(x))$ for all $x \in M$.

In [4] López-Permouth and Malik show that $R$-fzmod is a top category over $R$-mod. In fact, they define a contravariant functor $s : R\text{-mod} \to \mathbf{Cord}$ in the following way. For a left $R$-module $M$, let $s(M) = \{\mu : M \to [0, 1] \mid (M, \mu) \text{ is a fuzzy left } R\text{-module}\}$. Take a left $R$-module homomorphism $f : M \to N$. The map $s(f) : s(N) \to s(M)$ is defined by $s(f)(\mu) = \mu_f$ for all $\mu \in S(N)$ where $\mu_f(x) = \mu(f(x))$ for all $x \in M$. Note that the corresponding (see Remark 1(b)) covariant functor $t : R\text{-mod} \to \mathbf{Cord}$ is defined on modules $M$ in $R$-mod by $t(M) = s(M)$ and on arrows $f : M \to N$ in $R$-mod by $t(f)(\mu) = \mu^f$ for all $\mu \in t(M)$ where $\mu^f(y) = \sup_{x \in M}\{\mu(x) \mid f(x) = y\}$. for all $y \in N$.

**Proposition 5** (López-Permouth and Malik [4])**.** *Let $(M, \mu)$ be a fuzzy module.*

(a) *$(M, \mu)$ is projective if and only if $M$ is projective and $\mu = \mu_0$, where $\mu_0(x) = 0$ for all $x \in M \setminus \{0\}$.*
(b) *$(M, \mu)$ is injective if and only if $M$ is injective and $\mu = \mu_1$, where $\mu_1(x) = 1$ for all $x \in M$.*

*Proof.* This follows directly from Proposition 2. □

**Proposition 6** (López-Permouth and Malik [4])**.** *The category $R$-fzmod is additive, complete and cocomplete. However, it is not abelian.*

*Proof.* Since $R$-mod is an additive, complete and cocomplete category, by Proposition 3 and Proposition 4(b), the same holds for $R$-fzmod. Since any nonzero module in $R$-mod always can be equipped with the two different grade functions $\mu_0$ and $\mu_1$ (see Proposition 5), we get, by Proposition 4(c), that $R$-fzmod is not abelian. □

## 5. Fuzzy composition graphs

In this section, we introduce fuzzy composition graphs (see Definition 4). Our goal is to present categories of fuzzy composition graphs as top categories (see Proposition 9). The idea is to adapt the construction in Section 4 of the contravariant functor $s$ to our situation.

**Definition 4.** A fuzzy composition graph is a pair $(A, \mu)$ where $A$ is a composition graph $\mu$ is a grade function on $A$, that is, a map $\mu : \mathrm{ar}(A) \to [0, 1]$ satisfying $\mu(\alpha\beta) \geq \min(\mu(\alpha), \mu(\beta))$ for all arrows $\alpha$ and $\beta$ in $A$ with $\mathrm{d}(\alpha) = \mathrm{c}(\beta)$. An arrow of fuzzy composition graphs $(A, \mu) \to (A', \mu')$ is an arrow of composition graphs $f : A \to A'$ satisfying $\mu(\alpha) \leq \mu'(f(\alpha))$ for all arrows $\alpha$ in $A$. If $\mathbf{A}$ is a category of composition graphs, then we say that the fuzzified category of **A**, denoted $\mathbf{A}^{\mathrm{fz}}$, has as objects all fuzzy composition graphs $(A,\mu)$ where $A$ is a composition graph in **A** and as arrows the arrows of fuzzy composition graphs $(A,\mu) \to (A',\mu')$ where $A$ and $A'$ are composition graphs in **A**.

Assume for the rest of the article that **A** is a category of small composition graphs. If $\mu$ and $\mu'$ are grade functions on a composition graph $A$, then we put $\mu \leq \mu'$ if $\mu(\alpha) \leq \mu'(\alpha)$ for all arrows $\alpha$ in $A$. For a family $\mu_i$, $i \in I$, of grade functions on $A$, define the grade functions $\wedge_{i\in I}\mu_i$ and $\vee_{i\in I}\mu_i$ on $A$ by

$$\wedge_{i\in I}\mu_i(\alpha) = \inf\{\mu_i(\alpha) \mid i \in I\}$$

and

$$\vee_{i\in I}\mu_i(\alpha) = \inf\{\mu(\alpha) \mid \mu \text{ is a grade function on } A \text{ and } \mu_i \leq \mu \text{ for all } i \in I\}$$

for all arrows $\alpha$ in $A$. To show that $\wedge_{i\in I}\mu_i$ and $\vee_{i\in I}\mu_i$ really are grade functions, we need a well known lemma which we state without proof.

**Lemma 2.** *If $X$ and $Y$ are sets and $f$ is a real valued function on $X \times Y$, then $\inf_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\inf_{x\in X} f(x,y) = \inf_{(x,y)\in X\times Y} f(x,y)$.*

**Proposition 7.** *Let $A$ be a composition graph. The collection of grade functions on $A$ is a complete ordered lattice with least element $\mu_0$ and greatest element $\mu_1$, where $\mu_0(\alpha) = 0$ and $\mu_1(\alpha) = 1$ for all arrows $\alpha$ in $A$.*

*Proof.* Take arrows $\alpha$ and $\beta$ in $A$ with $\mathrm{d}(\alpha) = \mathrm{c}(\beta)$. By Lemma 2, we get that

$$\begin{aligned} \min(\wedge_{i\in I}\mu_i(\alpha), \wedge_{i\in I}\mu_i(\beta)) &= \inf_{i\in I}\min(\mu_i(\alpha), \mu_i(\beta)) \\ &\leq \inf_{i\in I}\mu_i(\alpha\beta) \\ &= \wedge_{i\in I}\mu_i(\alpha\beta) \end{aligned}$$

Let $\{\nu_j\}_{j\in J}$ denote the collection of grade functions on $A$ with the property that $\mu_i \leq \nu_j$ for all $i \in I$. Then, by Lemma 2 again, we get that

$$\begin{aligned} \min(\vee_{i\in I}\mu_i(\alpha), \vee_{i\in I}\mu_i(\beta)) &= \min(\inf_{j\in J}\nu_j(\alpha), \inf_{j\in J}\nu_j(\beta)) \\ &= \inf_{j\in J}\min(\nu_j(\alpha), \nu_j(\beta)) \\ &\leq \inf_{j\in J}\nu_j(\alpha\beta) \\ &= \vee_{i\in I}\mu_i(\alpha\beta) \end{aligned}$$

□

**Proposition 8.** *Let $f : A \to A'$ be an arrow of composition graphs and suppose that $\nu$ is a grade function on $A'$. If $\mu_i$, $i \in I$, is the collection of grade functions on $A$ with the property that $f$ is an arrow of fuzzy graphs $(A,\mu_i) \to (A',\nu)$, then $\vee_{i\in I}\mu_i = \nu_f$ where $\nu_f(\alpha) = \nu(f(\alpha))$ for all arrows $\alpha$ in $A$.*

*Proof.* First of all, $\nu_f$ is a grade function on $A$. In fact, if we take arrows $\alpha$ and $\beta$ in $A$ with $\mathrm{d}(\alpha) = \mathrm{c}(\beta)$, then

$$\begin{aligned} \min(\nu_f(\alpha), \nu_f(\beta)) &= \min(\nu(f(\alpha)), \nu(f(\beta))) \\ &\leq \nu(f(\alpha)f(\beta)) \\ &= \nu(f(\alpha\beta)) \\ &= \nu_f(\alpha\beta) \end{aligned}$$

By the definition of $\nu_f$, $f$ is an arrow of fuzzy graphs $(A, \nu_f) \to (A', \nu)$. Therefore, $\nu_f \leq \vee_{i\in I}\mu_i$. On the other hand, $f$ being an arrow of fuzzy graphs $(A, \mu_i) \to (A', \nu)$, $i \in I$, means precisely that $\mu_i \leq \nu_f$ for all $i \in I$. Therefore, $\vee_{i\in I}\mu_i \leq \nu_f$. □

Now we define a contravariant functor $s : \mathbf{A} \to \mathbf{Cord}$ in the following way. For a composition graph $A$ in $\mathbf{A}$, let $s(A)$ denote the collection of all grade functions on $A$. Given an arrow of composition graphs $f : A \to A'$ in $\mathbf{A}$ define $s(f) : s(A') \to s(A)$ by $s(f)(\nu) = \nu_f$ for all grade functions $\nu$ on $A'$.

**Proposition 9.** *The map $s$ is a contravariant functor $\mathbf{A} \to \mathbf{Cord}$ with the property that $s(f)$ respects infima for all arrows $f$ in $\mathbf{A}$. Therefore $\mathbf{A}^{\mathrm{fz}} = \mathbf{A}^{\mathrm{top}}$.*

*Proof.* First we show that $s(f)$ respects infinima. Take $\nu_i \in s(A')$, $i \in I$. Then

$$s(f)(\wedge_{i\in I}\nu_i)(\alpha) = (\wedge_{i\in I}\nu_i)_f(\alpha) = (\wedge_{i\in I}\nu_i)(f(\alpha)) = \inf_{i\in I}\{\nu_i(f(\alpha))\} =$$
$$= \inf_{i\in I}\{\nu_{if}(\alpha)\} = (\wedge_{i\in I}\nu_{if})(\alpha) = (\wedge_{i\in I}s(f)(\nu_i))(\alpha).$$

Next, we show that $s$ respects composition of arrows $f : A \to A'$ and $g : A' \to A''$ in $\mathbf{A}$. If $\mu \in s(A'')$ and $\alpha$ is an arrow in $A$, then

$$s(g \circ f)(\mu)(\alpha) = \mu((g \circ f)(\alpha)) = \mu(g(f(\alpha))) = \mu_g(f(\alpha)) = (\mu_g)_f(\alpha) =$$
$$= s(f)(\mu_g)(\alpha) = s(f)(s(g)(\mu))(\alpha) = (s(f) \circ s(g))(\mu)(\alpha).$$

Finally, it is easy to see that $s$ respects identity arrows. □

Now we wish to calculate the corresponding functor $t : \mathbf{A} \to \mathbf{Cord}$. By Remark 1(b) it is defined on objects $A$ in $\mathbf{A}$ by $t(A) = s(A)$ and on arrows $f : A \to A'$ by $t(f)(\mu) = \mu^f$ for all $\mu \in t(A)$, where $\mu^f = \wedge\nu$ and the infimum is taken over all $\nu \in t(A')$ with the property that $\mu \leq \nu_f$. In some cases this description of $t$ can be simplified. To show this, we need another well known lemma which we state without proof.

**Lemma 3.** *Let $Z$ be a subset of $\mathbb{R}^2$. Then*

$$\sup_{(x,y)\in Z}\min(x, y) \leq \min(\sup_{(x,y)\in Z}x, \sup_{(x,y)\in Z}y).$$

*If $Z = X \times Y$ for some subsets $X$ and $Y$ of $\mathbb{R}$, then equality holds.*

**Proposition 10.** *Suppose that $f : A \to A'$ is an arrow of composition graphs and that $\mu$ is a grade function on $A$. If $f_{ob} : ob(A) \to ob(A')$ is injective, then $\mu^f(\beta) = sup_{\alpha\in ar(A)}\{\mu(\alpha) \mid f(\alpha) = \beta\}$ for all arrows $\beta$ in $A'$.*

*Proof.* Put $\mu^0(\beta) := \sup_{\alpha\in ar(A)}\{\mu(\alpha) \mid f(\alpha) = \beta\}$ for all arrows $\beta$ in $A'$. We need to show that $\mu^0$ is a grade function on $A'$. Take arrows $\beta_1$ and $\beta_2$ in $A'$ with $\mathrm{d}(\beta_1) = \mathrm{c}(\beta_2)$. Then, since $f$ is injective on objects, we get that

$$\begin{array}{rcl} \mu^0(\beta_1\beta_2) & \geq & \sup\{\mu(\alpha_1\alpha_2) \mid \alpha_1, \alpha_2 \in \mathrm{ar}(A),\ f(\alpha_1) = \beta_1,\ f(\alpha_2) = \beta_2\} \\ & \geq & \sup_{\alpha_1,\alpha_2\in \mathrm{ar}(A),\ f(\alpha_1)=\beta_1,\ f(\alpha_2)=\beta_2} \min(\mu(\alpha_1), \mu(\alpha_2)) \end{array}$$

By using Lemma 3, we can therefore conclude that

$$\begin{array}{rcl} \mu^0(\beta_1\beta_2) & \geq & \min(\sup\{\mu(\alpha_1)\}_{\alpha_1\in \mathrm{ar}(A), f(\alpha_1)=\beta_1}, \sup\{\mu(\alpha_2)\}_{\alpha_2\in \mathrm{ar}(A), f(\alpha_2)=\beta_2}) \\ & = & \min(\mu^0(\alpha_1), \mu^0(\alpha_2)) \end{array}$$

□

**Corollary 1.** *Suppose that $f : A \to A'$ is an arrow of composition graphs and that $\mu$ is a grade function on $A$. If $A$ has one object, then $\mu^f(\beta) = sup_{\alpha\in ar(A)}\{\mu(\alpha) \mid f(\alpha) = \beta\}$ for all arrows $\beta$ in $A'$.*

*Proof.* This follows directly from Proposition 10. □

**Remark 2.** (a) Proposition 10 generalizes a result of Rosenfeld [9] from fuzzy groups to fuzzy composition graphs. Also compare with Proposition 4 of Anthony and Sherwood [1].

(b) We say that a grade function $\mu$ on a composition graph $A$ is normalized (or invertible) if $\mu(\mathrm{id}_a) = 1$ (or $\mu(\alpha^{-1}) = \mu(\alpha)$) for all $a \in \mathrm{ob}(A)$ (or all invertible arrows $\alpha$ in $A$). An arrow of fuzzy composition graphs $(A, \mu) \to (A', \mu')$ with $\mu$ and $\mu'$ normalized (or invertible) is an arrow of fuzzy composition graphs respecting the identity elements (or the invertible elements). If $\mathbf{A}$ is a category of composition graphs, then we say that the normalized (or invertible) fuzzified category of $\mathbf{A}$, denoted $\mathbf{A}^{\mathrm{fz}_0}$ (or $\mathbf{A}^{\mathrm{fz}_i}$), has as objects all normalized (or invertible) fuzzy composition graphs $(A, \mu)$ where $A$ is a composition graph in $\mathbf{A}$ and as arrows the arrows of normalized (or invertible) fuzzy composition graphs $(A, \mu) \to (A', \mu')$ where $A$ and $A'$ are composition graphs in $\mathbf{A}$. Analogously, the normalized invertible category $\mathbf{A}^{\mathrm{fz}_{0,i}}$ of $\mathbf{A}$ is defined. By slightly modifying the results and some of the definitions in this section it is easy to see that all the categories $\mathbf{A}^{\mathrm{fz}_0}$, $\mathbf{A}^{\mathrm{fz}_i}$ and $\mathbf{A}^{\mathrm{fz}_{0,i}}$ are top categories over $\mathbf{A}$. We leave the details to the reader.

(c) It is easy to see that the condition that $f$ is injective on objects from Proposition 10 can not be omitted in general. In fact, let $A$ be the composition graph with two objects $i$ and $j$ and one arrow $\alpha : i \to j$. Also let $A'$ be the cyclic group of order two generated by $\beta$, with one object identified with 1. If we define grade functions $\mu$ and $\mu'$ on $A$ and $A'$ respectively by $\mu(\alpha) = \mu'(\beta) = \mu'(\beta) = 1$ and let $f : A \to A'$ be defined on objects by $f(i) = f(j) = 1$ and on arrows by $f(\alpha) = \beta$, then it is easy to see that $0 = \mu^0(1) = \mu^0(\beta \cdot \beta) < 1 = \min(\mu^0(\beta), \mu^0(\beta))$. Therefore, $\mu^0$ is not a grade function on $A'$ and hence can not be equal to $\mu^f$.

## 6. Properties of fuzzy composition graphs

In this section, we apply the results from Section 3 and Section 5 on fuzzy composition graphs. We begin by investigating when arrows of fuzzy composition graphs are monic, epic and isomorphisms (see Proposition 11). Then we determine when fuzzy composition graphs are projective and injective (see Proposition 12). Thereafter we show when various types of limits and colimits exist in categories of fuzzy composition graphs (see Proposition 13 and Corollaries 2-6). In the end of the section we determine when categories of fuzzy composition graphs are Ab-categories, additive categories or abelian categories (see Corollary 7).

**Proposition 11.** *Let $f : (A, \mu) \to (A', \mu')$ be an arrow in $\mathbf{A}^{\mathrm{fz}}$. Let $\underline{f}$ denote the underlying map of composition graphs $A \to A'$ in $\mathbf{A}$. Then $f$ is monic (epic, an isomorphism) if and only if $\underline{f}$ is monic (epic, an isomorphism).*

*Proof.* This follows directly from Proposition 1. □

**Proposition 12.** *An object $(A, \mu)$ in $\mathbf{A}^{\mathrm{fz}}$ is projective (or injective) if and only if $A$ is projective (or injective) in $\mathbf{A}$ and $\mu = \mu_0$ (or $\mu = \mu_1$).*

*Proof.* This follows directly from Proposition 2. □

**Proposition 13.** *Let $\mathbf{I}$ be small category. Assume that $F : \mathbf{I} \to \mathbf{A}^{\mathrm{fz}}$ is a functor with $F(i) = (A_i, \mu_i)$, $i \in \mathrm{ob}(\mathbf{I})$, for some fuzzy composition graphs $(A_i, \mu_i)$ in $\mathbf{A}^{\mathrm{fz}}$.*

- (a) *A limit of $F$ exists if and only if a limit of $\underline{F}$ exists. In that case a limit of $F$ is given by $(\varprojlim A_i, \underline{\mu})$ where $\underline{\mu}(\alpha) = \inf\{\mu_i(f_i(\alpha)) \mid i \in \mathbf{S}\}$ for all $\alpha \in \mathrm{ar} \varprojlim A_i$ where $f_i : \varprojlim F_i \to F_i$ are the arrows associated with the choice of limit of $\underline{F}$ and $\mathbf{S}$ is any sufficient subcategory of $\mathbf{I}$. In particular, $\mathbf{A}^{\mathrm{fz}}$ is complete if and only if $\mathbf{A}$ is complete.*
- (b) *A colimit of $F$ exists if and only if a colimit of $\underline{F}$ exists. In that case a colimit of $F$ is given by $(\varinjlim A_i, \overline{\mu})$ where $\overline{\mu}(\alpha) = \inf_{\mu}\mu(\alpha)$ where $\mu$ runs over all grade functions on $\varinjlim A_i$ such that $\mu(\alpha) \geq \mu_i(\alpha_i)$ for all $i \in I$ and all $\alpha_i \in A_i$ such that $f'_i(\alpha_i) = \alpha$, and $f'_i : F_i \to \varinjlim F_i$ are the arrows associated with the choice of colimit of $\underline{F}$ and $\mathbf{S}$ is any cosufficient subcategory of $\mathbf{I}$. In particular, $\mathbf{A}^{\mathrm{fz}}$ is cocomplete if and only if $\mathbf{A}$ is cocomplete.*

*Proof.* This follows directly from Proposition 3. □

Now we will specialize Proposition 13 to various types of limits and colimits. We leave the proofs to the reader.

Recall that an object $a$ in a category $\mathbf{C}$ is called terminal (initial) if there for every object $b$ in the category is exactly one arrow $b \to a$ ($a \to b$); $a$ is called a zero object if it is both terminal and initial. A terminal (initial) object is a limit (colimit) over the empty category.

**Corollary 2.** *With the above notations,*

(a) $\mathbf{A}^{\rm fz}$ *has a terminal (or initial) object if and only if* $\mathbf{A}$ *has a terminal (or initial) object. In that case a terminal (or initial) object of* $\mathbf{A}^{\rm fz}$ *is given by* $(A, \mu_1)$ *(or* $(A, \mu_0)$*) where* $A$ *is a terminal (or initial) object of* $\mathbf{A}$.

(b) *The object* $(A, \mu)$ *in* $\mathbf{A}^{\rm fz}$ *is a zero object if and only if* $A$ *is a zero object in* $\mathbf{A}$ *and* $\mu$ *is the only grade function on* $A$.

Let $a_i$, $i \in I$, be a collection of objects in a category $\mathbf{C}$. Recall that an object $\prod_{i \in I} a_i$ (or $\coprod_{i \in I} a_i$) in $\mathbf{C}$ is called a product (or coproduct) of $a_i$, $i \in I$, if there are arrows $p_i : \prod_{i \in I} a_i \to a_i$ (or $q_i : a_i \to \coprod_{i \in I} a_i$), $i \in I$, with the property that for each object $b$ in $\mathbf{C}$ and each collection of arrows $p_i' : b \to a_i$ (or $q_i' : a_i \to b$), $i \in I$, there is a unique arrow $f : b \to \prod_{i \in I} a_i$ (or $g : \coprod_{i \in I} a_i \to b$) such that $p_i f = p_i'$ (or $g q_i = q_i'$) for all $i \in I$. A product (coproduct) is a limit (colimit) over a discrete category $\mathbf{I}$.

**Corollary 3.** *Let* $(A_i, \mu_i)$*,* $i \in I$*, be a set of fuzzy composition graphs in* $\mathbf{A}^{\rm fz}$.

(a) *The product of* $(A_i, \mu_i)$*,* $i \in I$*, exists in* $\mathbf{A}^{\rm fz}$ *if and only if the product of* $A_i$*,* $i \in I$*, exists in* $\mathbf{A}$*. In that case, if we let* $p_i : \prod_{i \in I} A_i \to A_i$*,* $i \in I$*, denote the corresponding arrows in* $\mathbf{A}$*, then* $(\prod_{i \in I} A_i, \underline{\mu})$ *is a product of* $(A_i, \mu_i)$*, where* $\underline{\mu}(\alpha) = \inf_{i \in I} \mu_i(p_i(\alpha))$ *for all arrows* $\alpha$ *in* $\prod_{i \in I} A_i$*. In particular, (finite) products exist in* $\mathbf{A}^{\rm fz}$ *if and only they exist in* $\mathbf{A}$.

(b) *The coproduct of* $(A_i, \mu_i)$*,* $i \in I$*, exists in* $\mathbf{A}^{\rm fz}$ *if and only if the coproduct of* $A_i$*,* $i \in I$*, exists in* $\mathbf{A}$*. In that case, if we let* $q_i : A_i \to \coprod_{i \in I} A_i$*,* $i \in I$*, denote the corresponding arrows in* $\mathbf{A}$*, then* $(\coprod_{i \in I} A_i, \overline{\mu})$ *is a coproduct of* $(A_i, \mu_i)$*,* $i \in I$*, where* $\overline{\mu}(\alpha) = \inf_\mu \mu(\alpha)$ *where* $\mu$ *runs over all grade functions on* $\coprod A_i$ *such that* $\mu(\alpha) \geq \mu_i(\alpha_i)$ *for all* $i \in I$ *all* $\alpha_i \in A_i$ *such that* $q_i(\alpha_i) = \alpha$*. In particular, (finite) coproducts exist in* $\mathbf{A}^{\rm fz}$ *if and only they exist in* $\mathbf{A}$.

Let $f : a \to c$ (or $f : c \to a$) and $g : b \to c$ (or $g : c \to b$) be arrows in a category $\mathbf{C}$. Recall that an object $a \times_c b$ (or $a \times^c b$) is called a pullback (or pushout) of $f$ and $g$ if there are arrows $p : a \times_c b \to a$ (or $p : a \to a \times^c b$) and $q : a \times_c b \to b$ (or $q : b \to a \times^c b$) such that $fp = gq$ (or $pf = qg$) with the following property. For each pair of arrows $h : d \to a$ (or $h : a \to d$) and $k : d \to b$ (or $k : b \to d$) with $fh = gk$ (or $hf = kg$) there is a unique arrow $r : d \to a \times_c b$ (or $r : a \times^c b \to d$) such that $pr = h$ (or $rp = h$) and $qr = k$ (or $rq = k$). Pullbacks (pushouts) are limits (colimits) over the category $\mathbf{I}$ with three objects 1, 2 and 3, and two non identity arrows $1 \to 2 \leftarrow 3$.

**Corollary 4.** *With the above notations,*

(a) *Let* $f : (A, \mu) \to (C, \tau)$ *and* $g : (B, \nu) \to (C, \tau)$ *be arrows in* $\mathbf{A}^{\rm fz}$*. Then the pullback of* $f$ *and* $g$ *exists in* $\mathbf{A}^{\rm fz}$ *if and only if the corresponding pullback exists in* $\mathbf{A}$*. In that case, if* $p : A \times_C B \to A$ *and* $q : A \times_C B \to B$ *are the arrows defining the pullback in* $\mathbf{A}$, *then a pullback of $f$ and $g$ is given by $(A \times_C B, \underline{\mu})$ where $\underline{\mu}(\alpha) = \min(\mu(p(\alpha)), \nu(q(\alpha)))$ for all arrows $\alpha$ in $A \times_C B$. In particular, pullbacks exist in $\mathbf{A}^{\mathrm{fz}}$ if and only if they exist in $\mathbf{A}$.*

(b) *Let $f : (C, \tau) \to (A, \mu)$ and $g : (C, \tau) \to (B, \nu)$ be arrows in $\mathbf{A}^{\mathrm{fz}}$. Then the pushout of $f$ and $g$ exists in $\mathbf{C}^{\mathrm{top}}$ if and only if the corresponding pushout exists in $\mathbf{A}$. In that case, if $p : a \to a \times_c b$ and $q : b \to a \times_c b$ are the arrows defining the pushout in $\mathbf{A}$, then a pushout of $f$ and $g$ is given by $(A \times^C B, \overline{\mu})$ where $\overline{\mu}(\alpha) = \inf_\mu \mu(\alpha)$ where $\mu$ runs over all grade functions on $A \times^C B$ such that $\mu(\alpha) \geq \max(\mu(\alpha'), \nu(\alpha''))$ for all arrows $\alpha'$ and $\alpha''$ in $A$ and $B$ respectively with the property that $p(\alpha') = q(\alpha'') = \alpha$. In particular, pushouts exist in $\mathbf{A}^{\mathrm{fz}}$ if and only if they exist in $\mathbf{A}$.*

Recall that an equalizer (or coequalizer) of a pair of arrows $\alpha, \beta : a \to b$ in a category $\mathbf{C}$ is an arrow $\gamma : c \to a$ (or $\gamma : b \to c$) in $\mathbf{C}$ such that $\alpha\gamma = \beta\gamma$ (or $\gamma\alpha = \gamma\beta$) and such that for any arrow $\delta : d \to a$ (or $\delta : d \to b$) with the property that $\alpha\delta = \beta\delta$ (or $\delta\alpha = \delta\beta$), then there is a unique arrow $\epsilon : c \to d$ with $\delta = \gamma\epsilon$ (or $\gamma = \epsilon\delta$). Equalizers (coequalizers) are limits (colimits) over the category with two objects 1 and 2 and two non identity arrows $1 \rightrightarrows 2$.

**Corollary 5.** *Let $f$ and $g$ be arrows $(A, \mu) \to (B, \nu)$ in $\mathbf{A}^{\mathrm{fz}}$.*

(a) *An equalizer of $f$ and $g$ exists in $\mathbf{A}^{\mathrm{fz}}$ if and only if the corresponding equalizer exists in $\mathbf{A}$. In that case, if $C$ is the corresponding equalizer in $\mathbf{A}$ and $h : C \to A$ is the induced arrow, then an equalizer of $f$ and $g$ is given by $(C, \underline{\mu})$ where $\underline{\mu}(\alpha) = \mu(h(\alpha))$ for all arrows $\alpha$ in $C$. In particular, equalizers exist in $\mathbf{A}^{\mathrm{fz}}$ if and only if $\mathbf{A}$ has equalizers.*

(b) *A coequalizer of $f$ and $g$ exists in $\mathbf{A}^{\mathrm{fz}}$ if the corresponding coequalizer exists in $\mathbf{A}$. In that case, if $C$ is the corresponding coequalizer in $\mathbf{A}$ and $h : B \to C$ is the induced arrow, then a coequalizer of $f$ and $g$ is given by $(C, \overline{\mu})$ where $\overline{\mu}(\alpha) = \inf_\tau \tau(\alpha)$ where $\tau$ runs over all grade functions on $C$ such that $\tau(\alpha) \geq \sup\{\nu(\alpha') \mid h(\alpha') = \alpha\}$ for all arrows $\alpha$ in $C$. In particular, coequalizers exist in $\mathbf{A}^{\mathrm{fz}}$ if and only if they exist in $\mathbf{A}$.*

Let $\mathbf{C}$ be a category with a null object. Recall that a kernel (or cokernel) of an arrow $\alpha : a \to b$ in $\mathbf{C}$ is an equalizer (or coequalizer) of $\alpha$ and the zero arrow $0 : a \to b$.

**Corollary 6.** *Let $f : (A, \mu) \to (B, \nu)$ be an arrow in $\mathbf{A}^{\mathrm{fz}}$.*

(a) *A kernel of $f$ exists in $\mathbf{A}^{\mathrm{fz}}$ if and only if the corresponding kernel exists in $\mathbf{A}$. In that case, a kernel of $f$ is given by $(K, \underline{\mu})$ where $i : K \to A$ is the corresponding kernel in $\mathbf{A}$ and $\underline{\mu}(\alpha) = \mu(i(\alpha))$ for all arrows $\alpha$ in $K$. In particular, kernels exist in $\mathbf{A}^{\mathrm{fz}}$ if and only if kernels exist in $\mathbf{A}$.*

(b) *A cokernel of $f$ exists in $\mathbf{A}^{\mathrm{fz}}$ if and only if a the corresponding cokernel exists in $\mathbf{A}$. In that case, a cokernel of $f$ is given by $(C, \overline{\mu})$*

*where $p : B \to C$ is the corresponding cokernel in* $\mathbf{A}$ *and* $\overline{\mu}(\alpha) = \inf_\tau \tau(\alpha)$ *where $\tau$ runs over all grade functions on $C$ such that* $\tau(\alpha) \geq \sup\{\nu(\alpha') \mid p(\alpha') = \alpha\}$*. In particular, cokernels exist in* $\mathbf{A}^{\mathrm{fz}}$ *if and only if cokernels exist in* $\mathbf{A}$*.*

**Corollary 7.** *Let* $\mathbf{A}$ *be an Ab-category. Assume that*

$$\mu((f + f')(\alpha)) \geq \min(\mu(f(\alpha)), \mu(f'(\alpha)))$$

*for all arrows $f, f' : A \to B$ in* $\mathbf{A}$*, all grade functions $\mu$ on $B$ and all arrows $\alpha$ in* $\mathbf{A}$*.*

(a) $\mathbf{A}^{\mathrm{fz}}$ *is an Ab-category.*
(b) $\mathbf{A}^{\mathrm{fz}}$ *is additive if and only if* $\mathbf{A}$ *is additive.*
(c) $\mathbf{A}^{\mathrm{fz}}$ *is abelian if and only if* $\mathbf{A}$ *is abelian and all objects in* $\mathbf{A}$ *only have one grade function.*

*Proof.* This follows directly from Proposition 4. □

**Remark 3.** (a) All of the results in this section can be adjusted to cover the categories $\mathbf{A}^{\mathrm{fz}_0}$, $\mathbf{A}^{\mathrm{fz}_i}$ and $\mathbf{A}^{\mathrm{fz}_{0,i}}$ from Remark 2(b). We leave the details to the reader.

(b) If $\mathbf{A}$ is a category of one object composition graphs, then, by Corollary 1, the description of many of the colimits in this section can be simplified. We leave the details of this analysis also to the reader.

## 7. Applications

In this section, we exemplify the results from Section 6 on the (normalized/invertible) fuzzified versions of the following categories of graphs with composition: **Set**, **Cat**, **Grd**, **Mon**, **Grp** and **Ab**. All index categories $\mathbf{I}$ in this section are assumed to be small.

We begin with fuzzy sets. First of all, note that $\mathbf{Set}^{\mathrm{fz}_0} = \mathbf{Set}$ so this gives nothing new. Also, $\mathbf{Set}^{\mathrm{fz}_i} = \mathbf{Set}^{\mathrm{fz}}$ so we will only discuss the ordinary fuzzy sets, namely $\mathbf{Set}^{\mathrm{fz}}$. By Proposition 11, the monic, epic and isomorphism arrows in $\mathbf{Set}^{\mathrm{fz}}$ are precisely the fuzzy arrows whose underlying arrows in **Set** are injective, surjective and bijective. The category **Set** is complete and cocomplete. In fact, let $F : \mathbf{I} \to \mathbf{Set}$ be a functor. A limit of $F$ is then given by the set $A$ of $(a_i)_{i \in \mathrm{ob}(\mathbf{I})}$ in $\prod_{i \in \mathrm{ob}(\mathbf{I})} F(i)$ with the property that $p_j(a_j) = F(\alpha)p_i(a_i)$ for all $\alpha : i \to j$ where $p_i : \prod_{i \in \mathrm{ob}(\mathbf{I})} F(i) \to F(i)$ denotes the $i$th projection. A colimit of $F$ is given by the set $B := \uplus_{i \in \mathrm{ob}(\mathbf{I})} F(i) / \sim$ where $\sim$ is the equivalence relation generated by the relation $R$ on $\uplus_{i \in \mathrm{ob}(\mathbf{I})} F(i)$ defined by letting $b_i R b_j$, for $b_i \in F(i)$ and $b_j \in F(j)$, if there is $\alpha : i \to j$ such that $b_i = F(\alpha) b_j$. Here $\uplus$ denotes the disjoint union. By Proposition 13, the category $\mathbf{Set}^{\mathrm{fz}}$ of fuzzy sets is therefore also complete and cocomplete. Now assume that $\overline{F} : \mathbf{I} \to \mathbf{Set}^{\mathrm{fz}}$ is a functor that lifts $F$. Then $\overline{F}(i) = (F_i, \mu_i)$, $i \in \mathrm{ob}\mathbf{I}$, for some fuzzy sets $\mu_i : F_i \to [0,1]$. By Proposition 13(a) and the above notation, a limit of $\overline{F}$ is given by $(A, \underline{\mu})$ where $\underline{\mu}$ is the restriction to $A$ of the map $\prod_{i \in \mathrm{ob}\mathbf{I}} A_i \ni (a_i)_{i \in I} \mapsto \inf\{\mu_i(a_i)\}_{i \in I}$. To calculate colimits in $\mathbf{Set}^{\mathrm{fz}}$, we note first the simplification $\vee_{i\in I}\mu_i(\cdot) = \sup_{i\in I}\mu_i(\cdot)$. By Proposition 10 and Proposition 13(b), a colimit of $F$ is now given by $(B, \overline{\mu})$ where $\overline{\mu}(a) = \sup_{a',i}\mu_i(a')$ where $i \in I$ and $a' \in A_i$ are chosen so that the image of $a'$ in $A$ equals $a$. Furthermore, all objects in **Set** are injective and projective. Hence, by Proposition 12, the projective and injective objects in $\mathbf{Set}^{\mathrm{fz}}$ are given by $(A, \mu_0)$ and $(A, \mu_1)$ respectively for some set $A$. The empty set and the sets with cardinality one are the initial and terminal objects respectively in **Set**. Hence, by Corollary 2(a), the initial and terminal objects of $\mathbf{Set}^{\mathrm{fz}}$ are given by $(\emptyset, \mu_0)$ and $(\{a\}, \mu_1)$ respectively. Let $\mu_i : A_i \to [0,1]$, $i \in I$, be a family of fuzzy sets. By Corollary 3, their product is given by $(\prod_{i\in I} A_i, \underline{\mu})$ where $\underline{\mu}((a_i)_{i\in I}) := \inf\{\mu_i(a_i)\}_{i\in I}$ for all $(a_i)_{i\in I} \in \prod_{i\in I} A_i$. Their coproduct is given by $(\uplus_{i\in I}A_i, \overline{\mu})$ where $\overline{\mu}(a) := \mu_i(a)$ if $a \in A_i$. Let $\mu : A \to [0,1]$, $\nu : B \to [0,1]$ and $\tau : C \to [0,1]$ be fuzzy sets. Assume that $f : (A,\mu) \to (C,\tau)$ and $g : (B,\nu) \to (C,\tau)$ are fuzzy maps. By Proposition Proposition 4(a), a pullback of $f$ and $g$ is given by $(D,\rho)$ where $D = \{(a,b) \in A \times B \mid f(a) = g(b)\}$ and $\rho(a,b) = \min(\mu(a),\nu(b))$ for all $(a,b) \in A \times B$. Now assume that $f : (C,\tau) \to (A,\mu)$ and $g : (C,\tau) \to (B,\nu)$ are fuzzy maps. By Proposition 4(b), a pushout of $f$ and $g$ is given by $(E,\epsilon)$ where $E = A \uplus B/\sim$ where $\sim$ is the equivalence relation generated by the identification of $f(c)$ and $g(c)$ for all $c \in C$. Also, $\epsilon(d) = \sup_{a,b}\max(\mu(a),\nu(b))$ where the supremum is taken over $a \in A$ and $b \in B$ so that their image in $E$ equals $d$. Let $f$ and $g$ be fuzzy maps $(A,\mu) \to (B,\nu)$. Then the equalizer of $f$ and $g$ is $(C, \mu|_C)$ where $C = \{a \in A \mid f(a) = g(a)\}$. The coequalizer of $f$ and $g$ is $(B/\sim, \nu')$ where $\sim$ is the equivalence relation on $B$ generated by the identification of $f(a)$ and $g(a)$ for all $a \in A$. The grade function $\mu' : B/\sim \to [0,1]$ is defined by $\mu'(d) = \sup\{\nu(b) \mid b \in B,\ b \sim d\}$ for all $[d] \in B/\sim$. Since **Set** does not have any null objects, the same is true for $\mathbf{Set}^{\mathrm{fz}}$ (by Corollary 2). Hence, the concepts kernel and cokernel are meaningless to discuss. Of course (by Corollary 7 or a simple direct argument) $\mathbf{Set}^{\mathrm{fz}}$ is not an abelian category.

Next, we analyse the category of fuzzy small categories. We will only consider $\mathbf{Cat}^{\mathrm{fz}}$. Since the same arguments work in the normalized (or invertible) case, we leave the details in that analysis to the reader. The category **Cat** is complete and cocomplete. Hence, by Proposition 13, the same is true for $\mathbf{Cat}^{\mathrm{fz}}$. Since the construction of limits in **Cat** is a slight extension of the construction described above for **Set** and the construction of colimits in **Cat** is a bit complicated (see e.g. [5]), we will only restrict ourselves to the description of products and coproducts in $\mathbf{Cat}^{\mathrm{fz}}$. Let $\mathbf{A}_i$, $i \in I$, be a set of categories. Recall that a product $\mathbf{A}$ of these categories has as objects the product $\prod_{i\in I} \mathrm{ob}(\mathbf{A}_i)$ and as arrows $\prod_{i\in I} \mathrm{ar}(\mathbf{A}_i)$. A coproduct $\mathbf{B}$ is the category with objects $\uplus_{i\in I}\mathrm{ob}(\mathbf{A}_i)$ and as arrows $\uplus_{i\in I}\mathrm{ar}(\mathbf{A}_i)$. Now, if $(\mathbf{A}_i, \mu_i)$, $i \in I$, is a set of fuzzy categories, then, by Corollary 3(a), a product of these categories is given by $(\mathbf{A}, \underline{\mu})$ where $\underline{\mu}((\alpha_i)_{i\in I}) := \inf\{\mu_i(\alpha_i)\}_{i\in I}$ for all $(\alpha_i)_{i\in I} \in \prod_{i\in I} \mathrm{ar}(\mathbf{A}_i)$. By Corollary 3(b), a coproduct of these fuzzy categories is given by $(\mathbf{B}, \overline{\mu})$ where $\overline{\mu}(\alpha) = \mu_i(\alpha)$ if $\alpha \in \mathrm{ar}(\mathbf{A}_i)$.

Now we discuss fuzzy groupoids, monoids and (abelian) groups. Since the categories **Grd**, **Mon**, **Grp** and **Ab** are complete and cocomplete, the same is true for the categories $\mathbf{Grd}^{\mathrm{fz}}$, $\mathbf{Mon}^{\mathrm{fz}}$, $\mathbf{Grp}^{\mathrm{fz}}$ and $\mathbf{Ab}^{\mathrm{fz}}$, by Proposition 13. The construction of limits in these categories is identical to the construction in **Cat**. The construction of colimits in **Grd** is also identical to the construction of colimits in **Cat** so we leave this also. As in the case with small categories above, we leave the normalized (or invertible) discussion to the reader. The colimits in **Mon** and **Grp** are defined implicitly (see e.g. p. 214 in [5]) by using the so called free product which we now describe. Let $G$ and $H$ be groups. Then their free product $G * H$ has as elements all finite words spelled in letters from $G$ and $H$. These words are multiplied by juxtaposition and equality is given by successive cancellations. Now suppose that $(G, \mu)$ and $(H, \nu)$ are fuzzy groups. By Proposition 10 and Corollary 3(b), their coproduct is given by $(G * H, \epsilon)$ where $\epsilon$ is defined in the following way. Take $x \in G * H$. If $x$ can not be simplified so that it belongs to either $G$ or $H$, then $\epsilon(x) = 0$. If $x$ can be simplified so that it equals either $y \in G$ or $z \in H$, then $\epsilon(x) = \max(\mu(y), \nu(z))$. We end the article with the well known colimit construction of a functor $F : \mathbf{I} \to \mathbf{Ab}$. It is given by the abelian group $A := \coprod_{i \in \mathrm{ob}(\mathbf{I})} F(i)/B$ where $B$ is the subgroup of $A$ generated by all expressions of the type $b_j - F(\alpha)(b_i)$ where $b_i \in F(i)$, $b_j \in F(j)$ and $\alpha : i \to j$. Now assume that $\overline{F} : \mathbf{I} \to \mathbf{Set}^{\mathrm{fz}}$ is a functor that lifts $F$. Then $\overline{F}(i) = (F_i, \mu_i)$, $i \in \mathrm{ob}(\mathbf{I})$, for some fuzzy sets $\mu_i : F_i \to [0, 1]$. By Proposition 10 and Proposition 13(b), a colimit of $F$ is now given by $(A, \overline{\mu})$ where $\overline{\mu}(a) = \sup_{a', i} \mu_i(a')$ where $i \in \mathrm{ob}(\mathbf{I})$ and the $a' \in A_i$ are chosen so that the image of $a'$ in $A$ equals $a$.

**Remark 4.** Classically, the categories of fuzzy monoids, fuzzy groups, fuzzy abelian groups coincide (in our notation) with $\mathbf{Mon}^{\mathrm{fz}_0}$, $\mathbf{Grp}^{\mathrm{fz}_{0,i}}$ and $\mathbf{Grp}^{\mathrm{fz}_{0,i}}$ respectively. It seems to the author that fuzzy groupids have not previously been studied (or even defined) in the literature.

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